Due date: Wednesday, July 17

Short Answer
State the answer to each of the following questions. It is not necessary to write down any justification (though of course one would want to be able to explain how each solution was obtained).

1. Define a function \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = x \) for \( x \leq 0 \) and \( f(x) = x^2 \) for \( x > 0 \). Use Theorem 4.1 (which relates differentiability to sequence convergence) to show that \( f \) is not differentiable at \( x_0 = 0 \). That is, find two sequences \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) in the domain of \( f \) which both converge to \( x_0 = 0 \) but for which the sequences \( \{T(x_n)\}_{n=1}^{\infty} \) and \( \{T(y_n)\}_{n=1}^{\infty} \) do not converge to the same value.

2. Define \( f : [0, 2] \to \mathbb{R} \) by \( f(x) = \sqrt{2x - x^2} \). Show that \( f \) satisfies the conditions of Rolle’s Theorem, and find \( c \) such that \( f'(c) = 0 \). (Assume for this problem that we know rules for computing basic derivatives.)

Proofs
For the remaining problems, a proof is required. In particular, a complete solution must be stated in sentence form with appropriate justification for each step.

1. Use the definition of differentiability to show that the function \( f : \{x \in \mathbb{R} : x \geq 0\} \to \mathbb{R} \) given by \( f(x) = \sqrt{x} \) is differentiable at every point in its domain except \( x = 0 \).

2. Prove that the definition of the derivative and the alternate definition of the derivative given in the text are the same.

3. Suppose \( f : [a, b] \to [c, d], \ g : [c, d] \to [p, q] \), and \( h : [p, q] \to \mathbb{R} \), with \( f \) differentiable at \( x_0 \in [a, b] \), \( g \) differentiable at \( f(x_0) \), and \( h \) differentiable at \( g(f(x_0)) \). Prove that \( h \circ (g \circ f) \) is differentiable at \( x_0 \), and find an expression for the derivative.

4. Define \( f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}} \) for \( x \geq 0 \). Determine where \( f \) is differentiable, and compute its derivative.

5. Prove that the equation \( \cos x = x^3 + x^2 + 4x \) has exactly one real root in the interval \( [0, \pi/2] \). If it is helpful to you, you may assume we know that \( \cos(x) \) and \( \sin(x) \) are continuous and differentiable, and that we know their derivatives.

(Hint: First use one of the major theorems we’ve studied to show that there is at least one root in this interval, then use another of our major theorems to show that there is at most one root in this interval.)

6. Fix values of \( a, b \in \mathbb{R} \) such that \( 0 \leq a \leq b \), and take \( n \in \mathbb{N} \). Use the Mean Value Theorem to prove that \( na^{n-1}(b - a) \leq b^n - a^n \leq nb^{n-1}(b - a) \).