Math 4603 Advanced Calculus I
Homework 8: Sections 4.1-4.3

Due date: Wednesday, July 17

Short Answer
State the answer to each of the following questions. It is not necessary to write down any justification (though of course one would want to be able to explain how each solution was obtained).

1. Define a function \( f : R \rightarrow R \) by \( f(x) = x \) for \( x \leq 0 \) and \( f(x) = x^2 \) for \( x > 0 \). Use Theorem 4.1 (which relates differentiability to sequence convergence) to show that \( f \) is not differentiable at \( x_0 = 0 \). That is, find two sequences \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) in the domain of \( f \) which both converge to \( x_0 = 0 \) but for which the sequences \( \{T(x_n)\}_{n=1}^{\infty} \) and \( \{T(y_n)\}_{n=1}^{\infty} \) do not converge to the same value.

Solution: One possible solution is \( x_n = 1/n \) and \( y_n = -1/n \). The \( \{T(x_n)\}_{n=1}^{\infty} \) converges to 1 and \( \{T(y_n)\}_{n=1}^{\infty} \) converges to 0.

2. Define \( f : [0, 2] \rightarrow R \) by \( f(x) = \sqrt{2x - x^2} \). Show that \( f \) satisfies the conditions of Rolle’s Theorem, and find \( c \) such that \( f'(c) = 0 \). (Assume for this problem that we know rules for computing basic derivatives.)

Solution: \( f \) is continuous on \([0, 2]\) by Theorems 3.2 and 3.4, because \( 2x, -x^2 \), and \( \sqrt{x} \) are continuous on \([0, 2]\); \( f \) is differentiable on \((0, 2)\) by Theorems 4.3 and 4.4 because \( 2x, -x^2 \), and \( \sqrt{x} \) are differentiable on \((0, 2)\); \( f(0) = 0, f(2) = 0 \). \( f'(x) = (1/2)(2x - x^2)^{-1/2}(2 - 2x) = 0 \) when \( x = 1 \)

Proofs
For the remaining problems, a proof is required. In particular, a complete solution must be stated in sentence form with appropriate justification for each step.

1. Use the definition of differentiability to show that the function \( f : \{x \in R : x \geq 0\} \rightarrow R \) given by \( f(x) = \sqrt{x} \) is differentiable at every point in its domain except \( x = 0 \).

Solution: Fix some \( x_0 > 0 \). We will show that \( f(x) \) is differentiable at \( x_0 \), and therefore show that it is differentiable at every point in its domain except 0.

For \( x \geq 0 \) and \( x \neq x_0 \), define \( T(x) = \frac{f(x) - f(x_0)}{x - x_0} \). Then when \( x \geq 0 \) and \( x \neq x_0 \), we have

\[
T(x) = \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \frac{x - x_0}{(x - x_0)(\sqrt{x} + \sqrt{x_0})} = \frac{1}{\sqrt{x} + \sqrt{x_0}}.
\]

This shows that for \( x > 0 \) and \( x \neq x_0 \), \( T \) is a quotient of two functions. We would like to apply Theorem 2.4 which gives results for the algebra of limits. Note that the function in the numerator, 1, has a limit 1 at \( x_0 \). Also, the function in the denominator, \( \sqrt{x} + \sqrt{x_0} \), has a limit at \( x_0 \), \( 2\sqrt{x_0} \), and this limit is nonzero since \( x_0 \neq 0 \). In addition, the value of the function \( \sqrt{x} + \sqrt{x_0} \) is nonzero for all \( x \geq 0 \), that is, all \( x \) in the domain of the function \( f \), since \( \sqrt{x_0} > 0 \) and \( \sqrt{x} \geq 0 \).
Therefore, we may apply Theorem 2.4 (iii) to conclude that \( T(x) \) has a limit at \( x_0 \), and its limit is \( \frac{1}{2\sqrt{x_0}} \). Therefore, by the definition of the derivative of a function, \( f \) is differentiable at every \( x_0 > 0 \), and \( f(x_0) = \frac{1}{2\sqrt{x_0}} = (1/2)x_0^{-1/2} \).

2. Prove that the definition of the derivative and the alternate definition of the derivative given in the text are the same.

**Solution:** (\( \Rightarrow \)) Suppose that the function \( T(x) = \frac{f(x) - f(x_0)}{x-x_0} \), \( T : D \setminus \{x_0\} \to R \) has a limit at \( x_0 \).

Let \( \epsilon > 0 \) be given. Because \( T \) has a limit at \( x_0 \), there exists \( \delta \) such that if

\[
0 < |x - x_0| < \delta \quad \text{and} \quad x \in D,
\]

then

\[
\left| \frac{f(x) - f(x_0)}{x-x_0} - L \right| < \epsilon
\]

for some limit \( L \).

Let \( t \) be such that \( x_0 + t \in D \setminus \{x_0\} \), and suppose \( 0 < |t - 0| < \delta \). Write \( x = x_0 + t \), so \( 0 < |x - x_0| < \delta \). This means that

\[
\left| \frac{f(x_0 + t) - f(x_0)}{t} - L \right| = \left| \frac{f(x) - f(x_0)}{x-x_0} - L \right| < \epsilon.
\]

(\( \Leftarrow \)) Conversely, suppose that the function \( Q(t) = \frac{f(x_0+t) - f(x_0)}{t} \) defined on \( t \) where \( t + x_0 \in D \setminus \{x_0\} \), has a limit at 0.

Let \( \epsilon > 0 \) be given. Because \( Q \) has a limit at 0, there exists \( \delta \) such that if \( 0 < t < \delta \) and \( t + x_0 \in D \setminus \{x_0\} \), then

\[
\left| \frac{f(x_0+t) - f(x_0)}{t} - L \right| < \epsilon
\]

for some limit \( L \).

Let \( x \in D \setminus \{x_0\} \) with \( 0 < |x - x_0| < \delta \). Write \( t = x - x_0 \), so that \( 0 < |t| < \delta \). Then

\[
\left| \frac{f(x) - f(x_0)}{x-x_0} - L \right| = \left| \frac{f(x_0 + t) - f(x_0)}{t} - L \right| < \epsilon.
\]

This shows that the first definition of the derivative holds if and only if the second definition holds, so the two are equivalent.

3. Suppose \( f : [a, b] \to [c, d], g : [c, d] \to [p, q], \) and \( h : [p, q] \to R, \) with \( f \) differentiable at \( x_0 \in [a, b], g \) differentiable at \( f(x_0), \) and \( h \) differentiable at \( g(f(x_0)). \) Prove that \( h \circ (g \circ f) \) is differentiable at \( x_0, \) and find an expression for the derivative.

**Solution:** We will apply the Chain Rule twice.

First, note that \( f([a, b]) \subset [c, d] = \text{dom}(g). \) Since \( f \) is differentiable at \( x_0 \) and \( g \) is differentiable at \( f(x_0), \) then by the Chain Rule \( g \circ f \) is differentiable at \( x_0. \) We have \( (g \circ f)'(x_0) = g'(f(x_0))f'(x_0). \)

Next, note that \( (g \circ f)([a, b]) \subset [p, q] = \text{dom}(h). \) Since \( g \circ f \) is differentiable at \( x_0 \) and \( h \) is differentiable at \( (g \circ f)(x_0), \) by the Chain Rule we know that \( h \circ (g \circ f) \) is differentiable at \( x_0, \) and

\[
(h \circ (g \circ f))'(x_0) = h'(g(f(x_0)))g'(f(x_0))f'(x_0).
\]

4. Define \( f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}} \) for \( x \geq 0. \) Determine where \( f \) is differentiable, and compute its derivative.
Solution: The function \( g(x) = \sqrt{x} \) is differentiable on \((0, \infty)\) by the Proof 1. The function \( h(x) = x + \sqrt{x} \) is differentiable on \((0, \infty)\) by Theorem 4.3, since \( x \) and \( \sqrt{x} \) are both differentiable on this region. The function \( g(h(x)) = x + \sqrt{x} \) is differentiable \((0, \infty)\) by the Chain Rule, since \( h \) is differentiable \((0, \infty)\) and \( g \) is differentiable on \( h(0, \infty) = (0, \infty) \). The function \( j(x) = x + \sqrt{x} + \sqrt{x} \) is differentiable on \((0, \infty)\) by Theorem 4.3 because both \( x \) and \( h \) are differentiable on this region. Finally, the function \( g(j(x)) = x + \sqrt{x} + \sqrt{x} \) is differentiable on \((0, \infty)\) by the Chain Rule, since \( j \) is differentiable on this region and \( g \) is differentiable on \( j(0, \infty) = (0, \infty) \).

The value of this derivative is

\[
(1/2)(x + \sqrt{x})^{-1/2}(1 + (1/2)(x + \sqrt{x})^{-1/2}(1 + (1/2)x^{-1/2})).
\]

5. Prove that the equation \( \cos x = x^3 + x^2 + 4x \) has exactly one real root in the interval \([0, \pi/2]\). If it is helpful to you, you may assume we know that \( \cos(x) \) and \( \sin(x) \) are continuous and differentiable, and that we know their derivatives.

(Hint: First use one of the major theorems we’ve studied to show that there is at least one root in this interval, then use another of our major theorems to show that there is at most one root in this interval.)

Solution: Define \( f(x) = x^3 + x^2 + 4x - \cos x \). Observe that \( f \) is continuous because it is the sum of continuous functions. Also observe that \( f(0) = -1 < 0 \) and \( f(\pi/2) = (\pi/2)^3 + (\pi/2)^2 + 4/(\pi/2) > 0 \). By the Intermediate Value Theorem, there exists a point \( c \in (0, \pi/2) \) where \( f(c) = 0 \); that is, there exists a solution to \( \cos x = x^3 + x^2 + 4x \) in the interval \((0, \pi/2)\).

Now observe that \( f \) is not only continuous on \([0, \pi/2]\), but also \( f \) is differentiable on \((0, \pi/2)\) because it is the sum of functions which are differentiable on this interval. Suppose that there is more than one solution to this equation in the interval \([0, \pi/2]\); that is, suppose we have \( c, d \in [0, \pi/2] \) for which \( f(c) = 0 = f(d) \). By Rolle’s Theorem, there exists \( r \in (0, \pi/2) \) where \( f'(r) = 0 \). However, \( f'(x) = 3x^2 + 2x + 4 + \sin(x) \) is always positive on this interval, since for \( x \in [0, \pi/2] \) we have \( 3x^2 \geq 0, 2x \geq 0, 4 > 0, \) and \( \sin(x) \geq 0 \). This gives a contradiction, so there cannot be two points for which \( f(x) = 0 \). That is, the equation \( \cos x = x^3 + x^2 + 4x \) has at most one solution in the interval \([0, \pi/2]\).

Together, this shows that the given equation has exactly one solution in the interval \([0, \pi/2]\).

6. Fix values of \( a, b \in R \) such that \( 0 \leq a \leq b \), and take \( n \in J \). Use the Mean Value Theorem to prove that \( n(a^n - b^n - (b - a)) \leq (b^n - a^n) \leq nb^{n-1}(b - a) \).

Solution: First note that if \( a = b \), then \( b - a = 0 \) and \( b^n - a^n = 0 \), so equalities hold in the desired statement.

Now suppose that \( a < b \). Consider the function \( f(x) = x^n \), which we know to be continuous on \([a, b]\) and differentiable on \((a, b)\). By the Mean Value Theorem, there exists \( c \in (a, b) \) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \). That is, there is some \( c \) with \( a < c < b \) where \( nc^{n-1} = \frac{b^n - a^n}{b - a} \). But since \( a < c < b \), we have
\( na^{n-1} < nc^{n-1} < nb^{n-1} \), and therefore

\[ na^{n-1} < \frac{b^n - a^n}{b - a} < nb^{n-1} \]

\[ na^{n-1}(b - a) < b^n - a^n < nb^{n-1}(b - a), \]

and so the desired statement holds.