1. Introduction, motivation, etc.

**Lecture 1** For most of this lecture, we will be motivating why we care about Lie algebras and why they are defined the way they are. If there are sections of it that don’t make sense, they can be skipped (although if none of it makes sense, you should be worried).

Connected Lie groups are groups which are also differentiable manifolds. The simple algebraic groups over algebraically closed fields (which is a nice class of Lie groups) are:

\[
SL(n) = \{ A \in \text{Mat}_n | \det A = 1 \}
\]

\[
SO(n) = \{ A \in SL_n | AA^T = I \}
\]

\[
SP(n) = \{ A \in SL_n | \text{some symplectic form} \}
\]

and exactly 5 others (as we will actually see!)

(We are ignoring finite centers, so \(\text{Spin}(n)\), the double cover of \(SO(n)\), is missing.)

**Remark.** Each of these groups has a maximal compact subgroup. For example, \(SU(n)\) is the maximal compact subgroup of \(SL(n)\), while \(SO_n(\mathbb{R})\) is the maximal compact subgroup of \(SO_n(\mathbb{C})\). These are themselves Lie groups, of course. The representations of the maximal compact subgroup are the same as the algebraic representations of the simple algebraic group are the same as the finite-dimensional representations of its (the simple algebraic group’s) Lie algebra – which are what we study.

Lie groups complicated objects: for example, \(SU_2(\mathbb{C})\) is homotopic to a 3-sphere, and \(SU_n\) is homotopic to a twisted product of a 3-sphere, a 5-sphere, a 7-sphere, etc. Thus, studying Lie groups requires some amount of algebraic topology and a lot of algebraic geometry. We want to replace these “complicated” disciplines by the “easy” discipline of linear algebra.

Therefore, instead of a Lie group \(G\) we will be considering \(\mathfrak{g} = T_1G\), the tangent space to \(G\) at the identity.

**Definition.** A *linear algebraic group* is a subgroup of some \(GL_n\) defined by polynomial equations in the matrix coefficients.

We see that the examples above are linear algebraic groups.
Remark. It is not altogether obvious that this is an intrinsic characterisation (it would be nice not to depend on the embedding into $GL_n$). The intrinsic characterisation is as “affine algebraic groups,” that is, groups which are affine algebraic varieties and for which multiplication and inverse are morphisms of affine algebraic varieties. One direction of this identification is relatively easy (we just need to check that multiplication and inverse really are morphisms); in the other direction, we need to show that any affine algebraic group has a faithful finite-dimensional representation, i.e. embeds in $GL(V)$. This involves looking at the ring of functions and doing something with it.

We will now explain a tangent space if you haven’t met with it before.

Example 1.1. Let’s pretend we’re physicists, since we’re in their building anyway. Let $G = SL_2$, and let $\epsilon \ll 1$. Then for

$$g = \begin{pmatrix} 1 & \epsilon a \\ 1 & \epsilon d \end{pmatrix}$$

for $g$ to lie in $SL_2$, we must have $\det g = 1$, or

$$1 = \det \begin{pmatrix} 1 + \epsilon a & \epsilon b \\ \epsilon c & 1 + \epsilon d \end{pmatrix} + \text{h.o.t} = 1 + \epsilon(a + d) + \epsilon^2(\text{junk}) + \text{h.o.t}$$

Therefore, to have $g \in G$ we need to have $a + d = 0$.

To do this formally, we define the dual numbers

$$E = \mathbb{C}[\epsilon]/\epsilon^2 = \{a + b\epsilon | a, b \in \mathbb{C}\}$$

For a linear algebraic group $G$, we consider

$$G(E) = \{A \in \text{Mat}_n(E) | A \text{ satisfies the polynomial equations defining } G \subset GL_n\}$$

For example,

$$SL_2(E) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} | \alpha, \beta, \gamma, \delta \in E, \alpha\delta - \beta\gamma = 1 \right\}$$

The natural map $E \to \mathbb{C}$, $\epsilon \mapsto 0$ gives a projection $\pi : G(E) \to G$. We define the Lie algebra of $G$ as

$$\mathfrak{g} \cong \pi^{-1}(I) \cong \{X \in \text{Mat}_n(\mathbb{C}) | I + \epsilon X \in G(E)\}.$$

In particular,

$$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}_2(\mathbb{C}) | a + d = 0 \right\}.$$

Remark. $I + X\epsilon$ represents an “infinitesimal change at $I$ in the direction $X$”. Equivalently, the germ of a curve $\text{Spec } \mathbb{C}[[\epsilon]] \to G$.

Exercise 1. (Do this only if you already know what a tangent space and tangent bundle are.) Show that $G(E) = TG$ is the tangent bundle to $G$, and $\mathfrak{g} = T_1G$ is the tangent space to $G$ at 1.

Example 1.2. Let $G = GL_n = \{A \in \text{Mat}_n | A^{-1} \text{exists}\}$.

Claim:

$$G(E) := \{\bar{A} \in \text{Mat}_n(E) | \bar{A}^{-1} \text{exists}\}$$

$$= \{A + B\epsilon | A, B \in \text{Mat}_n \mathbb{C}, A^{-1} \text{exists}\}$$

Indeed, $(A + B\epsilon)(A^{-1} - A^{-1}BA^{-1}\epsilon) = I$. 

Remark. It is not obvious that $GL_n$ is itself a linear algebraic group (what are the polynomial equations for “determinant is non-zero”?). However, we can think of $GL_n \subset \text{Mat}_{n+1}$ as \[
\begin{pmatrix} A \\ \lambda \end{pmatrix} \mid \det(A) \times \lambda = 1\].

**Example 1.3.** $G = SL_n \mathbb{C}$.

**Exercise 2.** $\det(I + \epsilon X) = 1 + \epsilon \cdot \text{trace}(X)$

As a corollary to the exercise, $\mathfrak{sl}_n = \{ X \in \text{Mat}_n \mid \text{trace}(X) = 0\}$.

**Example 1.4.** $G = O_n \mathbb{C} = \{ AA^T = I \}$. Then
\[
\mathfrak{g} := \{ X \in \text{Mat}_n \mathbb{C} \mid (I + \epsilon X)(I + \epsilon X)^T = I \} = \{ X \in \text{Mat}_n \mathbb{C} \mid I + \epsilon(X + X^T) = I \} = \{ X \in \text{Mat}_n \mathbb{C} \mid X + X^T = 0 \}
\]
the antisymmetric matrices. Now, if $X + X^T = 0$ then $2 \times \text{trace}(X) = 0$, and since we’re working over $\mathbb{C}$ and not in characteristic 2, we conclude $\text{trace}(X) = 0$. Therefore, this is also the Lie algebra of $SO_n$.

This is not terribly surprising, because topologically $O_n$ has two connected components corresponding to determinant $+1$ and $-1$ (they are images of each other via a reflection). Since $\mathfrak{g} = T_1G$, it cannot see the det $= -1$ component, so this is expected.

The above example prompts the question: what exactly is it in the structure of $\mathfrak{g}$ that we get from $G$ being a group?

The first thing to note is that $\mathfrak{g}$ does not get a multiplication. Indeed, $(I + A\epsilon)(I + B\epsilon) = I + (A + B)\epsilon$, which has nothing to do with multiplication.

The bilinear operation that turns out to generalize nicely is the commutator, $(P, Q) \mapsto PQP^{-1}Q^{-1}$. Taken as a map $G \times G \to G$ that sends $(I, I) \mapsto I$, this should give a map $T_1G \times T_1G \to T_1G$ by differentiation.

**Remark.** Generally, differentiation gives a **linear** map, whereas what we will get is a **bilinear** map. This is because we will in fact differentiate each coordinate: i.e. first differentiate the map $f_P : G \to G, Q \mapsto PQP^{-1}Q^{-1}$ with respect to $Q$ to get a map $\tilde{f}_P : \mathfrak{g} \to \mathfrak{g}$ (with $P \in G$ still) and then differentiate again to get a map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

**Exercise 3.** Show that $(PQP^{-1}Q^{-1})^{-1} = QPQ^{-1}P^{-1}$ implies $[A, B] = -[B, A]$, i.e. the bracket is skew-symmetric.

**Exercise 4.** Show that the associativity of multiplication in $G$ implies the **Jacobi identity**
\[
0 = [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]
\]
Remark. The meaning of “implies” in the above exercises is as follows: we want to think of the bracket as the derivative of the commutator, not as the explicit formula $[A, B] = AB - BA$ (which makes the skew-symmetry obvious, and the Jacobi identity only slightly less so). For example, we could have started working in a different category.

We will now define our object of study.

Definition. Let $K$ be a field, char $K \neq 2, 3$. A Lie algebra $\mathfrak{g}$ is a vector space over $K$ equipped with a bilinear map (Lie bracket) $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with the following properties:

1. $[X, Y] = -[Y, X]$ (skew-symmetry);
2. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity).

Lecture 2 Examples of the Lie algebra definition from last time:

Example 1.5. (1) $\mathfrak{gl}_n = \text{Mat}_n$ with $[A, B] = AB - BA$.
(2) $\mathfrak{so}_n = \{ A + A^T = 0 \}$
(3) $\mathfrak{sl}_n = \{ \text{trace } A = 0 \}$ (note that while trace $(AB) \neq 0$ for $A, B \in \mathfrak{sl}_n$, we do have trace $(AB) = \text{trace } (BA)$, so $[A, B] \in \mathfrak{sl}_n$ even though $AB \notin \mathfrak{sl}_n$)
(4) $\mathfrak{sp}_{2n} = \{ A \in \mathfrak{gl}_{2n} | JA^TJ^{-1} + A = 0 \}$ where $J$ is the symplectic form
\[
\begin{pmatrix}
1 & \cdots \\
\vdots & & -1 \\
-1 & & \ddots & \ddots \\
& & & & 1
\end{pmatrix}
\]
(5) $\mathfrak{b} = \left\{ \begin{pmatrix}
* & * & \cdots & * \\
* & * & \cdots & * \\
\ddots & & \ddots & \ddots \\
* & & & & *
\end{pmatrix} \in \mathfrak{gl}_n \right\}$ the upper triangular matrices (the name is $\mathfrak{b}$ for Borel, but we don’t need to worry about that yet)
(6) $\mathfrak{n} = \left\{ \begin{pmatrix}
0 & * & \cdots & * \\
0 & \cdots & * \\
\ddots & & \ddots & \ddots \\
0 & & & & 0
\end{pmatrix} \in \mathfrak{gl}_n \right\}$ the strictly upper triangular matrices (the name is $\mathfrak{n}$ for nilpotent, but we don’t need to worry about that yet)
(7) For any vector space $V$, let $[,] : V \times V \to V$ be the zero map. This is the “abelian Lie algebra”.

Exercise 5. (1) Check directly that $\mathfrak{gl}_n$ is a Lie algebra.
(2) Check that the other examples above are Lie subalgebras of $\mathfrak{gl}_n$, that is, vector subspaces closed under $[,]$.

Example 1.6. $\begin{pmatrix}
* & * \\
* & 0
\end{pmatrix}$ is not a Lie subalgebra of $\mathfrak{gl}_2$.

Exercise 6. Find the algebraic groups whose Lie algebras are given above.

Exercise 7. Classify all Lie algebras of dimension 3. (You might want to start with dimension 2. I’ll do dimension 1 for you on the board: skew symmetry of bracket means that bracket is zero, so there is only the abelian one.)
**Definition.** A *representation* of a Lie algebra \( g \) on a vector space \( V \) is a homomorphism of Lie algebras \( \phi : g \to gl_V \). We say that \( g \) acts on \( V \).

Observe that, above, our Lie algebras were defined with a (faithful) representation in tow. There is always a representation coming from the Lie algebra acting on itself (it is the derivative of the group acting on itself by conjugation):

**Definition.** For \( x \in g \) define \( \text{ad} (x) : g \to g \) as \( y \mapsto [x, y] \).

**Lemma 1.1.** \( \text{ad} : g \to \text{End} g \) is a representation (called the *adjoint representation*).

**Proof.** We need to check whether \( \text{ad} ([x, y]) = \text{ad} (x) \text{ad} (y) - \text{ad} (y) \text{ad} (x) \), i.e. whether the equality will hold when we apply it to a third element \( z \). But:

\[
\text{ad} ([x, y])(z) = [[x, y], z]
\]

and

\[
\text{ad} (x) \text{ad} (y)(z) - \text{ad} (y) \text{ad} (x)(z) = [x, [y, z]] - [y, [x, z]] = -[[y, z], x] - [[z, x], y].
\]

These are equal by the Jacobi identity. \( \Box \)

**Definition.** The *center* of \( g \) is \( \{ x \in g : [x, y] = 0, \forall y \in g \} = \ker(\text{ad} : g \to \text{End} g) \).

Thus, if \( g \) has no center, then \( \text{ad} \) is an embedding of \( g \) into \( \text{End} g \) (and conversely, of course).

Is it true that every finite-dimensional Lie algebra embeds into some \( gl_V \), i.e. is linear? (Equivalently, is it true that every finite-dimensional Lie algebra has a faithful finite-dimensional representation?) We see that if \( g \) has no center, then the adjoint representation makes this true. On the other hand, if we look inside \( gl_2 \), then \( n = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \) is abelian, so maps to zero in \( gl_n \), despite the fact that we started with \( n \subset gl_2 \). That is, we can’t always just take \( \text{ad} \).

**Theorem 1.2** (Ado’s theorem; we will not prove it). Any finite-dimensional Lie algebra over \( K \) is a subalgebra of \( gl_n \), i.e. admits a faithful finite-dimensional representation.

**Remark.** This is the Lie algebra equivalent of the statement we made last time about algebraic groups embedding into \( GL(V) \). That theorem was “easy” because there was a natural representation to look at (functions on the algebraic group). There isn’t such a natural object for Lie groups, so Ado’s theorem is actually hard.

**Example 1.7.** \( g = sl_2 \) with basis \( e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) (these are the standard names). Then \([e, f] = h, [h, e] = 2e, \) and \([h, f] = -2f\).

**Exercise 8.** Check this!

A representation of \( sl_2 \) is a triple \( E, F, H \in \text{Mat}_n \) satisfying the same bracket relations. Where might we find such a thing?

In this lecture and the next, we will get them as derivatives of representations of \( SL_2 \). We will then rederive them from just the linear algebra.

**Definition.** If \( G \) is an algebraic group, an *algebraic representation* of \( G \) on \( V \) is a homomorphism of groups \( \rho : G \to GL(V) \) defined by polynomial equations in the matrix coefficients.
(This can be done so that it is invariant of the embedding into $GL_n$. Think about it.)

To get a representation of the Lie algebra out of this, we again use the dual numbers $E$. If we use $E = K[\epsilon]/\epsilon^2$ instead of $K$, we will get a homomorphism of groups $G(E) \rightarrow GL_V(E)$. Moreover, since $\rho(I) = I$ and it commutes with the projection map, we get

$$\rho(I + Ae) = I + \epsilon \times (\text{some function of } A) = I + \epsilon d\rho(A)$$

(this is to be taken as the definition of $d\rho(A)$).

Exercise 9. $d\rho$ is the derivative of $\rho$ evaluated at $I$ (i.e., $d\rho : T_I G \rightarrow T_I GL_V$).

Exercise 10. The fact that $\rho : G \rightarrow GL_V$ was a group homomorphism means that $d\rho : \mathfrak{g} \rightarrow \mathfrak{gl}_V$ is a Lie algebra homomorphism, i.e. $V$ is a representation of $\mathfrak{g}$.

Example 1.8. $G = SL_2$.

Let $L(n)$ be the space of homogeneous polynomials of degree $n$ in two variables $x,y$, with basis $x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n$ (so dim $L(n) = n + 1$). Then $GL_2$ acts on $L(n)$ by change of coordinates: for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $f \in L(n)$ we have

$$((\rho_n g)(x, y)) = f(ax + cy, bx + dy)$$

In particular, $\rho_0$ is the trivial representation of $GL_2$, $\rho_1$ is the usual 2-dimensional representation $K^2$, and

$$\rho_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

Since $GL_2$ acts on $L(n)$, we see that $SL_2$ acts on $L(n)$.

Remark. The proper way to think of this is as follows. $GL_2$ acts on $\mathbb{P}^1$ and on $\mathcal{O}(n)$ on $\mathbb{P}^1$, hence on the global sections $\Gamma(\mathcal{O}(n), \mathbb{P}^1) = S^n K^2$. That’s the source of these representations. We can do this sort of thing for all the other algebraic groups we listed previously, using flag varieties instead of $\mathbb{P}^1$ and possibly higher (co)homologies instead of the global sections (this is a theorem of Borel, Weil, and Bott). That, however, requires algebraic geometry, and gets harder to do in infinitely many dimensions.

Differentiating the above representations, e.g. for $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we see

$$\rho(I + \epsilon e)x^iy^j = x^i(\epsilon x + y)^j = x^iy^j + \epsilon jx^{i+1}y^{j-1}$$

Therefore, $d\rho(e)x^iy^j = jx^{i+1}y^{j-1}$,

Exercise 11. (1) In the action of the Lie algebra,

$$e(x^iy^j) = jx^{i+1}y^{j-1}$$

$$f(x^iy^j) = ix^{i-1}y^{j+1}$$

$$h(x^iy^j) = (i - j)x^iy^j$$

(2) Check directly that these formulae give a representation of $\mathfrak{sl}_2$.

(3) Check that $L(2)$ is the adjoint representation.

(4) Show that the formulae $e = x\frac{\partial}{\partial y}, \ f = y\frac{\partial}{\partial x}, \ h = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ give an (infinite-dimensional!) representation of $\mathfrak{sl}_2$ on $k[x, y]$. 
(5) Let char \((k) = 0\). Show that \(L(n)\) is an irreducible representation of \(\mathfrak{sl}_2\), hence of \(SL_2\).

**Lecture 3** Last time we defined a functor

(algebraic representations of a linear algebraic group \(G\)) →  

(Lie algebra representations of \(\mathfrak{g} = \text{Lie}(G)\))

via \(\rho \mapsto d\rho\).

This functor is not as nice as you would like to believe, except for the Lie algebras we care about.

**Example 1.9.** \(G = \mathbb{C}^\times \implies \mathfrak{g} = \text{Lie}(G) = \mathbb{C}\) with \([x, y] = 0\).

A representation of \(\mathfrak{g} = \mathbb{C}\) on \(V\) is a matrix \(A \in \text{End}(V)\) (to \(\rho : \mathbb{C} \to \text{End}(V)\) corresponds \(A = \rho(1)\)).

\(W \subseteq V\) is a submodule iff \(AW \subseteq W\), and \(\rho\) is isomorphic to \(\rho' : \mathfrak{g} \to \text{End}(V')\) iff \(A\) and \(A'\) are conjugate as matrices.

Therefore, representations of \(\mathfrak{g}\) correspond to the Jordan normal form of matrices.

As any linear transformation over \(\mathbb{C}\) has an eigenvector, there’s always a 1D subrep of \(V\). Therefore, \(V\) is irreducible iff \(\dim V = 1\). Also, \(V\) is completely decomposable (a direct sum of irreducible representations) iff \(A\) is diagonalizable.

For example, if \(A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ \vdots & \ddots & \ddots \\ \vdots & \ddots & 1 \\ 0 & \end{pmatrix}\) then the associated representation is indecomposable, but not irreducible. Invariant subspaces are \(<e_1>, <e_1, e_2>, \ldots, <e_1, e_2, \ldots, e_n>\), and none of them has an invariant orthogonal complement.

Now let’s look at the representations of \(G = \mathbb{C}^\times\).

The irreducible representations are \(\rho_n : G \to GL_1 = \text{Aut}(\mathbb{C})\) via \(z \mapsto (\text{multiplication by})\ z^n\) for \(n \in \mathbb{Z}\). Every finite-dimensional representation is a direct sum of these.

**Remark.** Proving these statements about representations of \(G\) takes some theory of algebraic groups. However, you probably know this for \(S^1\), which is homotopic to \(G\). Recall that to get complete reducibility in the finite group setting you took an average over the group; you can do the same thing here, because \(S^1\) is compact and so has finite volume. In fact, for the theory that we study, our linear algebraic groups have maximal compact subgroups, which have the same representations, which for this reason are completely reducible. We will not prove that representations of the linear algebraic groups are the same as the representations of their maximal compact subgroups.

Observe that the representations of \(G\) and of \(\mathfrak{g}\) are not the same. Indeed, \(\rho \mapsto d\rho\) will send \(\rho_n \mapsto n \in \mathbb{C}\) (prove it!). Therefore, irreducible representations of \(G\) embed into irreducible representations of \(\mathfrak{g}\), but the map is not remotely surjective (not to mention the decomposability issue).

The above example isn’t very surprising, since \(\mathfrak{g}\) is also the Lie algebra \(\text{Lie}(\mathbb{C}, +)\), and we should not expect representations of \(\mathfrak{g}\) to coincide with those of \(\mathbb{C}^\times\). The surprising result is
Theorem 1.3 (Lie). $\rho \mapsto d\rho$ is an equivalence of categories $\text{Rep}G \to \text{Rep}g$ if $G$ is a simply connected simple algebraic group.

Remark. A “simple algebraic group” is not simple in the usual sense: e.g. $SL_n$ has a center, which is obviously a normal subgroup. However, if $G$ is simply connected and simple in the above sense, then the center of $G$ is finite (e.g. $Z_{SL_n} = \mu_n$, the $n$th roots of unity), and the normal subgroups are subgroups of the center.

Exercise 12. If $G$ is an algebraic group, and $Z$ is a finite central subgroup of $G$, then $\text{Lie}(G/Z) = \text{Lie}(G)$. (Morally, $Z$ identifies points “far away”, and therefore does not affect the tangent space at $I$.)

We have now seen that the map (algebraic groups) $\to$ (Lie algebras) is not injective (see above exercise, or – more shockingly – $G = C, C \times$). In fact,

Exercise 13. Let $G_n = C^\times \ltimes C$ where $C^\times$ acts on $C$ via $t \cdot \lambda = t^n\lambda$ (so $(t, \lambda)(t', \lambda') = (tt', t^n\lambda + \lambda')$. Show that $G_n \cong G_m$ iff $n = \pm m$. Show that $\text{Lie}(G_n) \cong Cx + Cy$ with $[x, y] = y$ independently of $n$.

The map also isn’t surjective (its image are the “algebraic Lie algebras”). This is easily seen in characteristic $p$; for example, $\mathfrak{s}\mathfrak{l}_p/\text{center}$ cannot be the image of an algebraic group. In general, algebraic groups have a Jordan decomposition – every element can be written as (semisimple) $\times$ (nilpotent), – and therefore the algebraic Lie algebras should have a Jordan decomposition as well. In Bourbaki, you can find an example of a 5-dimensional Lie algebra, for which the semisimple and the nilpotent elements lie only in the ambient $\mathfrak{g}\mathfrak{l}_V$ and not in the algebra as well.

2. Representations of $\mathfrak{s}\mathfrak{l}_2$

From now on, all algebras and representations are over $C$ (an algebraically closed field of characteristic 0). Periodically we’ll mention which results don’t need this.

Recall the basis of $\mathfrak{s}\mathfrak{l}_2$ was $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ with commutation relations $[e, f] = h, [h, e] = 2e, [h, f] = -2f$.

For the next while, we will be proving the following

Theorem 2.1.

1. For all $n \geq 0$ there exists a unique irreducible representation of $g = \mathfrak{s}\mathfrak{l}_2$ of dimension $n + 1$. (Recall $L(n)$ from the previous lecture; these are the only ones.)

2. Every finite-dimensional representation of $\mathfrak{s}\mathfrak{l}_2$ is a direct sum of irreducible representations.

Let $V$ be a representation of $\mathfrak{s}\mathfrak{l}_2$.

Definition. The $\lambda$-weight space for $V$ is $V_\lambda = \{ v \in V : hv = \lambda v \}$, the eigenvectors of $h$ with eigenvalue $\lambda$.

Example 2.1. In $L(n)$, we had $L(n)_{\lambda} = Cx^iy^j$ for $\lambda = i - j$.

Let $v \in V_\lambda$, and consider $ev$. We have

$$hev = (he - eh + eh)v = [h, e]v + e(hv) = 2ev + e\lambda v = (\lambda + 2)ev.$$ 

That is, if $v \in V_\lambda$ then $ev \in V_{\lambda+2}$ and similarly $fv \in V_{\lambda-2}$. (These are clearly iff.)
We will think of this pictorially as a chain
\[ \ldots \Rightarrow V_{\lambda-2} \Rightarrow V_{\lambda} \overset{e}{\Rightarrow} V_{\lambda+2} \Rightarrow \ldots \]

E.g., in \( L(n) \) we had \( x^iy^j \) sitting in the place of the \( V_{\lambda} \)'s, and the chain went from \( \lambda = n \) to \( \lambda = -n \):
\[
\begin{align*}
V_n & \iff V_{n-2} \iff V_{n-4} \iff \ldots \iff V_{-(n-4)} \iff V_{-(n-2)} \iff V_{-n} \\
\langle x^n \rangle & \iff \langle x^{n-2}y \rangle \iff \langle x^{n-4}y^2 \rangle \iff \ldots \iff \langle x^2y^{n-2} \rangle \iff \langle xy^{n-1} \rangle \iff \langle y^n \rangle
\end{align*}
\]
(Note that the string for \( L(n) \) has \( n+1 \) elements.)

**Definition.** If \( v \in V_{\lambda} \cap \ker e \), i.e. \( hv = \lambda v \) and \( ev = 0 \), then we say that \( v \) is a *highest-weight vector* of weight \( \lambda \).

**Lemma 2.2.** Let \( V \) be a representation of \( \mathfrak{sl}_2 \) and \( v \in V \) a highest-weight vector of weight \( \lambda \). Then \( W = \langle v, fv, f^2v, \ldots \rangle \) is an \( \mathfrak{sl}_2 \)-invariant subspace, i.e. a subrepresentation.

**Proof.** We need to show \( hw \subseteq W \), \( fw \subseteq W \), and \( eW \subseteq W \).

Note that \( fW \subseteq W \) by construction.

Further, since \( v \in V_{\lambda} \), we saw above that \( f^k v \in V_{\lambda-2k} \), and so \( hw \subseteq W \) as well.

Finally,
\[
\begin{align*}
ev & = 0 \\
efv & = (e+f)v = hv + f(0) = \lambda v \\
ef^2v & = (e+f)fv = (\lambda - 2)fv + f(\lambda v) = (2\lambda - 2)fv \\
ef^3v & = (e+f)f^2v = (\lambda - 4)f^2v + f(2\lambda - 2)fv = (3\lambda - 6)f^2v
\end{align*}
\]
and so on. \( \square \)

**Exercise 14.** \( ef^nv = n(\lambda - n + 1)f^{n-1}v \) for \( v \) a highest-weight vector of weight \( \lambda \).

**Lemma 2.3.** Let \( V \) be a finite-dimensional representation of \( \mathfrak{sl}_2 \) and \( v \in V \) a highest-weight vector of weight \( \lambda \). Then \( \lambda \in \mathbb{Z}_{\geq 0} \).

**Remark.** Somehow, the integrality condition is the Lie algebra remembering something about the topology of the group. (Saying what this “something” is more precisely is difficult.)

**Proof.** Note that \( f^k v \) all lie in different eigenspaces of \( h \), so if they are all nonzero then they are linearly independent. Since \( \dim V < \infty \), we conclude that there exists a \( k \) such that \( f^k v \neq 0 \) and \( f^{k+r} v = 0 \) for all \( r \geq 1 \). Then by the above exercise,
\[
0 = ef^{k+1}v = (k+1)(\lambda - k)f^kv
\]
from which \( \lambda = k \), a nonnegative integer. \( \square \)

**Proposition 2.4.** If \( V \) is a finite-dimensional representation of \( \mathfrak{sl}_2 \), it has a highest-weight vector.

**Proof.** We’re over \( \mathbb{C} \), so we can pick some eigenvector of \( h \). Now apply \( e \) to it repeatedly: \( v, ev, e^2v, \ldots \) belong to different eigenspaces of \( h \), so if they are nonzero, they are linearly independent. Therefore, there must be some \( k \) such that \( e^k v \neq 0 \) but \( e^{k+r} v = 0 \) for all \( r \geq 1 \). Then \( e^k v \) is a highest-weight vector of weight \( \lambda + 2k \). \( \square \)
Corollary 2.5. If $V$ is an irreducible representation of $\mathfrak{sl}_2$ then $\dim V = n + 1$, and $V$ has a basis $v_0, v_1, \ldots, v_n$ on which $\mathfrak{sl}_2$ acts as $hv_i = (n - 2i)v_i$; $fv_i = v_{i+1}$ (with $fv_n = 0$); and $ev_i = i(n - i + 1)v_{i-1}$. In particular, there exists a unique $(n + 1)$-dimensional irreducible representation, which must be isomorphic to $L(n)$.

Exercise 15. Work out how this basis is related to the basis we previously had for $L(n)$.

Lecture 4

We will now show that every finite-dimensional representation of $\mathfrak{sl}_2\mathbb{C}$ is a direct sum of irreducible representations, or (equivalently) the category of finite-dimensional representations of $\mathfrak{sl}_2\mathbb{C}$ is semisimple, or (equivalently) every representation of $\mathfrak{sl}_2\mathbb{C}$ is completely reducible.

Morally, we’ve shown that every such representation consists of “strings” descending from the highest weight, and we’ll now show that these strings don’t interact. We’ll first show that they don’t interact when they have different highest weights (easier) and then that they also don’t interact when they have the same highest weight (harder).

We will also show that $h$ acts diagonalisably on every finite-dimensional representation, while $e$ and $f$ are nilpotent. In fact,

Remark.

$$\text{Span } h = \{ \begin{pmatrix} a & \phantom{-} \phantom{-} \\ -a & \phantom{-} \phantom{-} \end{pmatrix} \} = \text{Lie} \{ \begin{pmatrix} t \\ t^{-1} \end{pmatrix} \} \subset \text{Lie } (SL_2)$$

where $\{ \begin{pmatrix} t \\ t^{-1} \end{pmatrix} \}$ is the maximal torus inside $SL_2$. Like for $\mathbb{C}^\times$, the representations here should correspond to integers. In some hazy way, the fact that we are picking out the representations of a circle and not any other 1-dimensional Lie algebra is a sign of the Lie algebra remembering something about the group. (Saying what it is remembering and how is a hard question.)

Example 2.2. Recall that $\mathbb{C}[x, y]$ is a representation of $\mathfrak{sl}_2$ via the action by differential operators, and is a direct sum of the representations $L(n)$.

Exercise 16. Show that the formulae $e = x \frac{\partial}{\partial y}$, $f = y \frac{\partial}{\partial x}$, $h = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$ give a representation of $\mathfrak{sl}_2$ on $x^\lambda y^\mu \mathbb{C}[x/y, y/x]$ for any $\lambda, \mu \in \mathbb{C}$, and describe the submodule structure.

Definition. Let $V$ be a finite-dimensional representation of $\mathfrak{sl}_2$. Define $\Omega = ef + fe + \frac{1}{2}h^2 \in \text{End } (V)$ to be the Casimir of $\mathfrak{sl}_2$.

Lemma 2.6. $\Omega$ is central; that is, $e\Omega = \Omega e$, $f\Omega = \Omega f$, and $h\Omega = \Omega h$.

Exercise 17. Prove it. For example,

$$e\Omega = e(ef + fe + \frac{1}{2}h^2) = e(ef - fe) + 2(efe) + \frac{1}{2}(eh - he)h + \frac{1}{2}heh$$

$$= 2efe + \frac{1}{2}heh = (fe - ef)e + 2efe + \frac{1}{2}h(he - eh) + \frac{1}{2}heh = \Omega e$$

Observe that we can write $\Omega = (ef - fe) + 2fe + \frac{1}{2}h^2 = (\frac{1}{2}h^2 + h) + 2fe$. This will be useful later.

Corollary 2.7. If $V$ is an irreducible representation of $\mathfrak{sl}_2$, then $\Omega$ acts on $V$ by a scalar.

Proof. Schur’s lemma. \[\square\]
Lemma 2.8. Let \( L(n) \) be the irreducible representation of \( \mathfrak{sl}_2 \) with highest weight \( n \). Then \( \Omega \) acts on \( L(n) \) by \( \frac{1}{2} n^2 + n \).

Proof. It suffices to check this on the highest-weight vector \( v \) (so \( hv = nv \) and \( ev = 0 \)). Then

\[
\Omega v = (\frac{1}{2} h^2 + h + 2fe)v = (\frac{1}{2} n^2 + n)v.
\]

Since \( \Omega \) commutes with \( f \) and \( L(n) \) is spanned by \( f^i v \), we could conclude directly (without using Schur’s lemma) that \( \Omega \) acts by this scalar. \( \square \)

Observe that if \( L(n) \) and \( L(m) \) are irreducible representations with different highest weights, then \( \Omega \) acts by a different scalar (since \( \frac{1}{2} n(n+2) \) is increasing in \( n \)).

Definition. Let \( V \) be a finite-dimensional representation of \( \mathfrak{sl}_2 \). Set

\[
V^\lambda = \{ v \in V : (\Omega - \lambda)^{\dim V} v = 0 \}.
\]

This is the generalised eigenspace of \( \Omega \) with eigenvalue \( \lambda \).

Claim. Each \( V^\lambda \) is a subrepresentation for \( \mathfrak{sl}_2 \).

Proof. Let \( x \in \mathfrak{sl}_2 \) and \( v \in V^\lambda \). Because \( \Omega \) is central, we have

\[
(\Omega - \lambda)^{\dim V} (xv) = x(\Omega - \lambda)^{\dim V} v = 0,
\]

so \( xv \in V^\lambda \) as well. \( \square \)

Now, if \( V^\lambda \neq 0 \), then \( \lambda = \frac{1}{2} n^2 + n \), and \( V^\lambda \) is somehow “glued together" from copies of \( L(n) \). More precisely:

Definition. Let \( W \) be a finite-dimensional \( g \)-module. A composition series for \( W \) is a sequence of submodules

\[
0 = W_0 \subseteq W_1 \subseteq \ldots \subseteq W_r = W
\]

such that \( W_i/W_{i-1} \) is an irreducible module.

Example 2.3. (1) \( g = \mathbb{C} \), \( W = \mathbb{C}^r \), where \( 1 \in \mathbb{C} \) acts as

\[
\begin{pmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

there exists a unique composition series for \( W \), namely,

\[
0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \ldots \subset \langle e_1, e_2, \ldots, e_r \rangle.
\]

(2) On the other hand, if \( g = \mathbb{C} \) and \( W = \mathbb{C} \) with the abelian action (i.e. \( 1 \in \mathbb{C} \) acts by \( 0 \)), then any chain \( 0 \subset W_1 \subset \ldots \subset W_r \) with \( \dim W_i = i \) will be a composition series.

Lemma 2.9. Composition series always exist.

Proof. We induct on the dimension of \( W \). Take an irreducible submodule \( W_1 \subseteq W \); then \( W/W_1 \) has smaller dimension, so has a composition series. Taking its preimage in \( W \) and sticking \( W_1 \) in front will do the trick. \( \square \)

Remark. The factors \( W_i/W_{i-1} \) are defined up to order.
So, the precise statement is that $V^\lambda$ has composition series with quotients $L(n)$ for some fixed $n$. Indeed, take an irreducible submodule $L(n) \subset V^\lambda$; note that $\Omega$ acts on $L(n)$ by $\frac{1}{2}n^2 + n$, so we must have $\lambda = \frac{1}{2}n^2 + n$, or in other words $n$ is uniquely determined by $\lambda$; and moreover, $\Omega$ acts on $V^\lambda/L(n)$, and its only generalised eigenvalue there is still $\lambda$, so we can repeat the argument.

**Claim.** $h$ acts on $V^\lambda$ with generalised eigenvalues in the set $\{n, n-2, \ldots, -n\}$.

**Proof.** This is a general fact about composition series. Let $h$ act on $W$ and let $W' \subset W$ be invariant under this action, i.e. $hW' \subseteq W$. Then

$$\{\text{generalised eigenvalues of } h \text{ on } W\} = \{\text{generalised eigenvalues of } h \text{ on } W'\}$$

$$\cup \{\text{generalised eigenvalues of } h \text{ on } W/W'\}.$$  

(You can see this by looking at the upper triangular matrix decomposition of $h$.) Now since $V^\lambda$ is composed of $L(n)$, the generalised eigenvalues of $h$ must lie in that set. □

Note also that on the kernel of $e : V^\lambda \to V^\lambda$ the only generalised eigenvalue of $h$ is $n$; that is, $(h - n)^{\dim V^\lambda} x = 0$ for $x \in V^\lambda \cap \ker e$. (This follows by applying the above observation to the composition series intersected with $\ker e$.)

**Lecture 5**

**Lemma 2.10.**

1. $hf^k = f^k(h - 2k)$
2. $ef^{n+1} = f^{n+1}e + (n + 1)f^n(h - n)$.

**Proof.** We saw (1) already.

**Exercise 18.** Prove (2) (by induction, e.g. for $n = 0$ the claim is $ef = fe + h$.) □

**Proposition 2.11.** $h$ acts diagonally on $\ker e : V^\lambda \to V^\lambda$; in fact, it acts as multiplication by $n$. That is,

$$\ker e : V^\lambda \to V^\lambda = (V^\lambda)_n = \{x \in V^\lambda : hx = nx\}.$$  

Recall we know that $\ker e$ is (in) the generalised eigenspace of $h$ with eigenvalue $n$ (and after the first line of the proof, we will have equality rather than just containment). We are showing that we can drop the word “generalised”.

**Proof.** If $hx = nx$ then $ex \in (V^\lambda)_{n+2} = 0$ (as generalised eigenvalues of $h$ on $V^\lambda$ are $n, n-2, \ldots, -n$) so $x \in \ker e$.

Conversely, let $x \in \ker e$; we know $(h - n)^{\dim V^\lambda} x = 0$. By the lemma above,

$$(h - n + 2k)^{\dim V^\lambda} f^k x = f^k (h - n)^{\dim V^\lambda} x = 0.$$  

That is, $f^k x$ belongs to the generalised eigenspace of $h$ with eigenvalue $h - n + 2k$.

On the other hand, for any $y \in \ker e$, $y \neq 0 \implies f^ny \neq 0$.

**Remark.** This should be an obvious property of upper-triangular matrices, but we’ll work it out in full detail.

Take $0 = W_0 \subset W_1 \subset \ldots \subset W_r = V^\lambda$ the composition series for $V^\lambda$ with quotients $L(n)$. There is an $i$ such that $y \in W_i$, $y \not\in W_{i-1}$. Let $\overline{y} = y + W_{i-1} \in W_i/W_{i-1} \cong L(n)$. Then $\overline{y}$ is the highest-weight vector in $L(n)$, and so $f^n\overline{y} \neq 0$ in $L(n)$. Therefore, $f^ny \neq 0$.
Now, $f^{n+1}x$ belongs to the generalised eigenspace of $h$ with eigenvalue $-n-2$, so is equal to 0 (as generalised eigenvalues of $h$ on $V^\lambda$ are $n, n-2, \ldots, -n$). Therefore, 

$$0 = e f^{n+1}x = (n+1)f^n(h-n)x + f^{n+1}ex = (n+1)f^n(h-n)x.$$ 

Now, $(h-n)x \in \ker e$ (it’s still in the generalised eigenspace of $h$ with eigenvalue $n$), so if $(h-n)x \neq 0$ then $f^n(h-n)x \neq 0$, which in characteristic 0 poses certain problems. Therefore, $hx = nx$. \hfill \square

We can now finish the proof of complete reducibility. Indeed, choose a basis $w_1, w_2, \ldots, w_k$ of $\ker e : V^\lambda \to V^\lambda$, and consider the “string” generated by each $w_i$; that is, consider 

$$\langle w_i, f w_i, f^2 w_i, \ldots, f^n w_i \rangle.$$ 

**Exercise 19.** Convince yourself that these strings constitute a direct sum decomposition of $V^\lambda$; that is, these are subrepresentations, nonintersecting, linearly independent, and span everything.

Therefore, $h$ acts semisimply and diagonalisably on $V^\lambda$, and therefore on all of $V$.

**Exercise 20.** Show that this is false in characteristic $p$. That is, show that irreducible representations of $\mathfrak{sl}_2(\mathbb{F}_p)$ are parametrised by $n \in \mathbb{N}$, and find a representation of $\mathfrak{sl}_2(\mathbb{F}_p)$ that does not decompose into a direct sum of irreducibles.

### 3. Consequences

Let $V$, $W$ be representations of a Lie algebra $\mathfrak{g}$.

**Claim.** $\mathfrak{g} \to \text{End}(V \otimes W) = \text{End}(V) \otimes \text{End}(W)$ via $x \mapsto x \otimes 1 + 1 \otimes x$ is a homomorphism of Lie algebras.

**Exercise 21.** Prove it. (This comes from differentiating the group homomorphism $G \to G \times G$ via $g \mapsto (g, g)$.)

**Corollary 3.1.** If $V$, $W$ are representations of $\mathfrak{g}$, then so is $V \otimes W$.

**Remark.** If $A$ is an algebra, $V$ and $W$ representations of $A$, then $V \otimes W$ is naturally a representation of $A \otimes A$. To make it a representation of $A$, we need an algebra homomorphism $A \to A \otimes A$. (An object with such a structure – plus some other properties – is called a Hopf algebra.) In some essential way, $\mathfrak{g}$ is a Hopf algebra. (Or, rather, a deformation of the universal enveloping algebra of $\mathfrak{g}$ is a Hopf algebra.)

Now, take $\mathfrak{g} = \mathfrak{sl}_2$. What is $L(n) \otimes L(m)$? That is, we know $L(n) \otimes L(m) = \oplus_k a_k L(k)$; what are the $a_k$?

One method of doing this would be to compute all the highest-weight vectors.

**Exercise 22.** Do this for $L(1) \otimes L(n)$. Then do it for $L(2) \otimes L(n)$.

As a start on the exercise, let $v_a$ denote the highest-weight vector in $L(a)$. Then $v_n \otimes v_m$ is a highest-weight vector in $L(n) \otimes L(m)$. Indeed,

$$h(v_n \otimes v_m) = (hv_n) \otimes v_m + v_n \otimes (hv_m) = (n+m)v_n \otimes v_m$$

$$e(v_n \otimes v_m) = (ev_n) \otimes v_m + v_n \otimes (ev_m) = 0 + 0 = 0$$

Therefore, 

$$L(n) \otimes L(m) = L(n+m) + \text{other stuff}$$
However, counting dimensions, we get

$$(n + 1)(m + 1) = (n + m + 1) + \dim(\text{other stuff}),$$

which shows that there is quite a lot of this “other stuff” in there.

One can write down explicit formulae for all the highest-weight vectors; these are complicated but mildly interesting. However, we don’t have to do it to determine the summands of $L(n) \otimes L(m)$.

Let $V$ be a finite-dimensional representation of $\mathfrak{sl}_2$.

**Definition.** The character of $V$ is $\text{ch} V = \sum_{n \in \mathbb{Z}} \dim V_n z^n \in \mathbb{N}[z, z^{-1}]$.

**Properties 3.2.**

1. $\text{ch} V|_{z=1} = \dim V$.

   This is equivalent to the claim that $h$ is diagonalisable with eigenvalues in $\mathbb{Z}$, so $V = \bigoplus_{n \in \mathbb{Z}} V_n$.

2. $\text{ch} L(n) = z^n + z^{n-2} + \ldots + z^{-n} = \frac{z^{n+1} + z^{-(n+1)}}{z + z^{-1}}$. Sometimes this is written as $[n + 1]_z$.

3. $\text{ch} V = \text{ch} W$ iff $V \cong W$.

   Note that $\text{ch} L(0) = 1, \text{ch} L(1) = z + z^{-1}, \text{ch} L(2) = z^2 + 1 + z^{-2}, \ldots$ form a $\mathbb{Z}$-basis for the space of symmetric Laurent polynomials with integer coefficients. On the other hand, by complete reducibility,

   $$V = \bigoplus_{n \geq 0} a_n L(n), \quad a_n \geq 0$$

   $$W = \bigoplus_{n \geq 0} b_n L(n), \quad b_n \geq 0$$

   $$V \cong W \iff a_n = b_n \text{ for all } n$$

   But since $\text{ch} L(n)$ form a basis, $\text{ch} V = \sum a_n \text{ch} L(n)$ determines $a_n$.

**Remark.** Note that for the representations, we only need nonnegative coefficients $a_n \geq 0$. On the other hand, if instead of looking at modules we were looking at chain complexes, the idea of $\mathbb{Z}$-linear combinations would become meaningful.

4. $\text{ch} (V \otimes W) = \text{ch} V \cdot \text{ch} W$

**Exercise 23** (Essential!). $V_n \otimes V_m \subseteq (V \otimes W)_{n+m}$. Therefore, $(V \otimes W)_p = \sum_{n+m=p} V_n \otimes V_m$. This is exactly how we multiply polynomials.

**Example 3.1.** We can now compute $L(1) \otimes L(3)$:

$$\text{ch } L(1) \cdot \text{ch } L(3) = (z + z^{-1})(z^3 + z + z^{-1} + z^{-3}) =$$

$$(z^4 + z^2 + 1 + z^{-2} + z^{-4}) + (z^2 + 1 + z^{-2}) = \text{ch } L(4) + \text{ch } L(2),$$

so $L(1) \otimes L(3) = L(4) \oplus L(2)$.

5. The Clebsch-Gordan rule:

$$L(n) \otimes L(m) = \bigoplus_{k = [n-m]}^{n+m} L(k).$$

There is a picture that goes with this rule (so you don’t actually need to remember it):
Figure 1. Crystal for $\mathfrak{sl}_2$.

The individual strings are the $L(a)$ in the decomposition.

One of the points of this course is that this sort of picture-drawing (called crystals) lets us decompose tensor products of representations of all other semisimple Lie algebras. (This fact was discovered about 20 years ago by Kashiwara and Lusztig.)

What we will actually show for the other semisimple Lie algebras:

- The category of finite-dimensional representations is completely reducible (i.e. all finite-dimensional representations are semisimple)
- We will parameterise the irreducible representations
- We will compute the character of the irreducible representations
- We will show how to take tensor products of irreducible representations by drawing pictures (crystals)

4. Structure and classification of simple Lie algebras

Lecture 6

We begin with some linear algebra preliminaries.

**Definition.** A Lie algebra $\mathfrak{g}$ is called *simple* if the only ideals in $\mathfrak{g}$ are 0 and $\mathfrak{g}$, and $\dim \mathfrak{g} > 0$ (i.e., $\mathfrak{g}$ is not abelian). A Lie algebra $\mathfrak{g}$ is called *semisimple* if it is a direct sum of simple Lie algebras.

**Remark.** Recall that $\mathbb{C}$ behaved quite differently from $\mathfrak{sl}_2$, so it is sensible to exclude the abelian Lie algebras.

**Definition.** The *derived algebra* of $\mathfrak{g}$ is $[\mathfrak{g}, \mathfrak{g}] = \text{Span} \{x, y : x, y \in \mathfrak{g}\}$.

**Exercise 24.**

1. $[\mathfrak{g}, \mathfrak{g}]$ is an ideal;
2. $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian.

**Definition.** The *central series* of $\mathfrak{g}$ is given by $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^n = [\mathfrak{g}^{n-1}, \mathfrak{g}]$, i.e.

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] \supseteq \ldots$$

The *derived series* of $\mathfrak{g}$ is given by $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$, i.e.

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \supseteq \ldots$$

**Remark.** Clearly, $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^n$.

**Definition.** $\mathfrak{g}$ is *nilpotent* if $\mathfrak{g}^n = 0$ for some $n > 0$, i.e. the central series terminates.
$\mathfrak{g}$ is *solvable* if $\mathfrak{g}^{(n)} = 0$ for some $n > 0$, i.e. the derived series terminates.
By the above remark, a nilpotent Lie algebra is necessarily solvable.

Remark. The term “solvable” comes from Galois theory (if the Galois group is solvable, the associated polynomial is solvable by radicals). We will see shortly where the term “nilpotent” comes from.

Example 4.1. (1) \( n = \begin{pmatrix} 0 & * & \ldots & * \\ 0 & \ldots & * \\ \vdots & \ddots & \ddots \\ 0 & \ldots & 0 \end{pmatrix} \in \mathfrak{gl}_n \) the strictly upper triangular matrices are a nilpotent Lie algebra.

(2) \( b = \begin{pmatrix} * & * & \ldots & * \\ * & \ldots & * \\ \vdots & \ddots & \ddots \\ * & \ldots & \ldots & * \end{pmatrix} \in \mathfrak{gl}_n \) the upper triangular matrices are a solvable Lie algebra.

Exercise 25 (Essential). (a) Compute the central and derived series to check that these algebras are nilpotent and solvable respectively.

(b) Compute the center of these algebras.

(3) Let \( W \) be a symplectic vector space with inner product \( \langle \cdot, \cdot \rangle \) (recall that a symplectic vector space is equipped with a nondegenerate bilinear antisymmetric form, and there is essentially one example: given a finite-dimensional vector space \( L \) let \( W = L + L^* \) with inner product \( \langle L, L \rangle = \langle L^*, L^* \rangle = 0 \) and \( \langle v, v^* \rangle = -\langle v^*, v \rangle = v^*(v) \)). Define the Heisenberg Lie algebra as

\[
\mathfrak{h}_W = W \oplus \mathbb{C} c \quad [w, w'] = \langle w, w' \rangle c, \quad [c, w] = 0.
\]

Exercise 26. Show this is a Lie algebra, and that it is nilpotent.

This is the most important nilpotent Lie algebra occurring in nature. For example, let \( L = \mathbb{C} \), then \( \mathfrak{h}_W = \mathbb{C} p + \mathbb{C} q + \mathbb{C} c \) with \([p, q] = c, [p, c] = [q, c] = 0 \) (as we know from classifying the 3-dimensional Lie algebras!) Show that this has a representation on \( \mathbb{C}[x] \) via \( q \mapsto \text{multiplication by } x \), \( p \mapsto \partial/\partial x \), \( c \mapsto 1 \).

(For a general vector space \( L \), the basis vectors \( v_1, \ldots, v_n \) map to multiplication by \( x_1, \ldots, x_n \), and their duals map to \( \partial/\partial x_1, \ldots, \partial/\partial x_n \)).

Properties 4.1. (1) Subalgebras and quotients of solvable Lie algebras are solvable. Subalgebras and quotients of nilpotent Lie algebras are nilpotent.

(2) Let \( g \) be a Lie algebra, and let \( h \) be an ideal of \( g \). Then \( g \) is solvable if and only if \( h \) and \( g/h \) are both solvable. (That is, solvable Lie algebras are built out of abelian Lie algebras, i.e. there is a refinement of the derived series such that the subquotients are 1-dimensional and therefore abelian.)

(3) \( g \) is nilpotent iff the center of \( g \) is nonzero, and \( g/(\text{center of } g) \) is nilpotent. Indeed, if \( g \) is nilpotent, then the central series is

\[
g \supseteq g^1 \supseteq \ldots \supseteq g^{n-1} \supseteq g^n = 0,
\]

and since \( g^n = [g^{n-1}, g] \) we must have \( g^{n-1} \) contained in the center of \( g \).

(4) \( g \) is nilpotent iff \( \text{ad } g \subseteq \mathfrak{gl}(g) \) is nilpotent, as we had an exact sequence

\[
0 \to \text{center of } g \to g \to \text{ad } g \to 0.
\]
Exercise 27. Prove the properties. (The proofs should all be easy and tedious.)

We now state a nontrivial result, which we will not prove.

**Theorem 4.2 (Lie’s Theorem).** Let \( g \subseteq \mathfrak{gl}(V) \) be a solvable Lie algebra, and let the base field \( k \) be algebraically closed and have characteristic zero. Then there exists a basis \( v_1, \ldots, v_n \) of \( V \) with respect to which \( g \subseteq \mathfrak{b}(V) \), i.e. the matrices of all elements of \( g \) are upper triangular.

Equivalently, there exists a linear function \( \lambda : g \to k \) and an element \( v \in V \), such that \( xv = \lambda(x)v \) for all \( x \in g \), i.e. \( v \) is a common eigenvector for all of \( g \), i.e. a one-dimensional subrepresentation.

In particular, the only irreducible finite-dimensional representations of \( g \) are one-dimensional.

Exercise 28. Show that these are actually equivalent. (One direction should be obvious; in the other, quotient by \( v_1 \) and repeat.)

Exercise 29. Show that we need \( k \) to be algebraically closed.

Exercise 30. Find a counterexample in characteristic \( p \).

*Proof.* Apply Lie’s theorem to the adjoint representation \( g \to \text{End}(g) \). With respect to some basis, \( \text{ad} g \subseteq \mathfrak{b}(g) \). Since \([\mathfrak{b}, \mathfrak{b}] \subseteq \mathfrak{n}\) (note the diagonal entries cancel out when we take the bracket), we see that \([\text{ad} g, \text{ad} g] \) is nilpotent. Since \( \text{ad} [g, g] = [\text{ad} g, \text{ad} g] \) (it’s a representation!), we see that \( \text{ad} [g, g] \) is nilpotent, and therefore so is \([g, g]\). \( \square \)

**Theorem 4.4 (Engel’s Theorem).** Let the base field \( k \) be arbitrary. \( g \) is nilpotent iff \( \text{ad} g \) consists of nilpotent endomorphisms of \( g \) (i.e., for all \( x \in g \), \( \text{ad} x \) is nilpotent).

Equivalently, if \( V \) is a finite-dimensional representation of \( g \) such that all elements of \( g \) act on \( V \) by nilpotent endomorphisms, then there exists \( v \in V \) such that \( x(v) = 0 \) for all \( x \in g \), i.e. \( V \) has the trivial representation as a subrep.

Equivalently, we can write \( x \) as a strictly upper triangular matrix.

Exercise 31. Show that these are actually equivalent.

**Lecture 7**

**Definition.** A symmetric bilinear form \( (, ) : g \times g \to k \) is invariant if \( ([x, y], z) = (x, [y, z]) \).

Exercise 32. If \( a \subseteq g \) is an ideal and \( (, ) \) is an invariant bilinear form, then \( a^\perp \) is an ideal.

**Definition.** If \( V \) is a finite-dimensional representation of \( g \), i.e. if \( \rho : g \to \mathfrak{gl}(V) \) is a homomorphism, define the trace form to be \( (x, y)_V = \text{trace} (\rho(x)\rho(y) : V \to V) \).

Exercise 33. Check that the trace form is symmetric, bilinear, and invariant. (They should all be obvious.)

**Example 4.2.** \( (, )_{\text{ad}} \) is the “Killing form” (named after a person, not after an action). This is the trace form attached to the adjoint representation. That is, \( (x, y)_{\text{ad}} = \text{trace} (\text{ad} x, \text{ad} y : g \to g) \).
Theorem 4.5 (Cartan’s criteria). Let \( g \subseteq \mathfrak{gl}(V) \) and let \( \text{char } k = 0 \). Then \( g \) is solvable if and only if for every \( x \in g \) and \( y \in [g, g] \) the trace form \((x, y)_V = 0\). (That is, \([g, g] \subseteq g^\perp\).)

Exercise 34. Lie’s theorem gives us one direction. Indeed, if \( g \) is solvable then we can take a basis in which \( x \) is an upper triangular matrix, \( y \) is strictly upper triangular, and then \( xy \) and \( yx \) both have zeros on the diagonal (and thus trace 0). That is, if \( g \) is solvable and nonabelian, all trace forms are degenerate. (The exercise is to convince yourself that this is true.)

Corollary 4.6. \( g \) is solvable iff \( ([g, g], [g, g])_{ad} = 0 \).

Proof. If \( g \) is solvable, Lie’s theorem gives the trace to be zero. Conversely, Cartan’s criteria tell us that \( ad g = g/(\text{center of } g) \) is solvable, and therefore so is \( g \). \( \square \)

Exercise 35. Not every invariant form is a trace form.

1. Construct a nondegenerate invariant form on \( \tilde{H} = \mathbb{C} \langle p, q, c, d \rangle \) where \([c, \tilde{H}] = 0, [p, q] = c, [d, p] = p, [d, q] = -q\).

2. Show \( \tilde{H} \) is solvable.

3. Extend the representation of \( \langle c, p, q \rangle = H \) on \( k[x] \) to a representation of \( \tilde{H} \).

Definition. The radical of \( g \), \( R(g) \), is the maximal solvable ideal in \( g \).

Exercise 36. (1) Show \( R(g) \) is the sum of all solvable ideals in \( g \) (i.e., show that the sum of solvable ideals is solvable).

(2) Show \( R(g/R(g)) = 0 \).

Theorem 4.7. In characteristic 0, the following are equivalent:

1. \( g \) is semisimple
2. \( R(g) = 0 \)
3. The Killing form \((,)_\text{ad}\) is nondegenerate (the Killing criterion).

Moreover, if \( g \) is semisimple, then every derivation \( D : g \to g \) is inner, but not conversely (where we define the terms “derivation” and “inner” below).

Definition. A derivation \( D : g \to g \) is a linear map satisfying \( D[x, y] = [Dx, y] + [x, Dy] \).

Example 4.3. \( \text{ad } (x) \) is a derivation for any \( x \). Derivations of the form \( \text{ad } (x) \) for some \( x \in g \) are called inner.

Remark. If \( g \) is any Lie algebra, we have the exact sequence

\[
0 \to R(g) \to g \to g/R(g) \to 0
\]

where \( R(g) \) is a solvable ideals, and \( g/R(g) \) is semisimple. That is, any Lie algebra has a maximal semisimple quotient, and the kernel is a solvable ideal. This shows you how much nicer the theory of Lie algebras is than the corresponding theory for finite groups. In particular, the corresponding statement for finite groups is essentially equivalent to the classification of the finite simple groups.

A theorem that we will neither prove nor use, but which is pretty:

Theorem 4.8 (Levi’s theorem). In characteristic 0, a stronger result is true. Namely, the exact sequence splits, i.e., there exists a subalgebra \( \mathcal{B} \subseteq g \) such that \( \mathcal{B} \cong g/R(g) \). (This subalgebra is not canonical; in particular, it is not an ideal of \( g \).) That is, \( g \) can be written as \( g = \mathcal{B} \ltimes R(g) \).
**Exercise 37.** Show that this fails in characteristic $p$. Let $\mathfrak{g} = \mathfrak{sl}_p(\mathbb{F}_p)$; show $R(\mathfrak{g}) = \mathbb{F}_p I$ (note that the identity matrix in dimension $p$ has trace $0$), but that there is no complement to $R(\mathfrak{g})$ that is an algebra.

**Proof of Theorem 4.7.** First, notice that $R(\mathfrak{g}) = 0$ if and only if $\mathfrak{g}$ has no nonzero abelian ideals. (In one direction, an abelian ideal is solvable; in the other, notice that the last term of the derived series is abelian.)

From now on, $(,)$ refers to the Killing form $(,)_ad$.

(3) $\implies$ (2): we show that if $a$ is an abelian ideal of $\mathfrak{g}$, then $a \subseteq \mathfrak{g}^\perp$. (Since the Killing form is nondegenerate, this means $a = 0$, and therefore $R(\mathfrak{g}) = 0$.)

Write $\mathfrak{g} = a + \mathfrak{h}$ where $\mathfrak{h}$ is a vector space complement to $a$ (not necessarily an ideal). For $a \in a$, $ad a$ has block matrix $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. For $x \in \mathfrak{g}$, $ad x$ has block matrix $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. (This is because $a$ is an ideal, so $[a, x] \in a$ for all $x \in \mathfrak{g}$, and moreover $ad a$ acts by $0$ on $a$.) Therefore,

$$\text{trace } (ad a \cdot ad x) = \text{trace } \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} = 0,$$

i.e. $(a, \mathfrak{g})_{ad} = 0$. Since the Killing form is nondegenerate, $a = 0$.

(2) $\implies$ (3): Let $r \subseteq \mathfrak{g}^\perp$ be an ideal (e.g. $r = \mathfrak{g}^\perp$), and suppose $r \neq 0$. Then $r \subseteq \mathfrak{gl}(\mathfrak{g})$ via $ad$, and $(x, y)_{ad} = 0$ for all $x, y \in r$. By Cartan’s criteria, $ad r = r/(\text{center of } r)$ is solvable, so $r$ is solvable, contradicting $R(\mathfrak{g}) = 0$.

**Exercise 38.** Show that $R(\mathfrak{g}) \supseteq \mathfrak{g}^\perp \supseteq [R(\mathfrak{g}), R(\mathfrak{g})]$.

(2), (3) $\implies$ (1): Let $(,)_ad$ be nondegenerate, and let $a \subseteq \mathfrak{g}$ be a minimal nonzero ideal.

We claim that $(,)_ad |a$ is either 0 or nondegenerate.

Indeed, the kernel of $(,)_ad |a = \{x \in a : (x, a)_ad = 0\} = a \cap a^\perp$ is an ideal!

Cartan’s criteria imply that $a$ is solvable if $(,)_ad |a$ is 0. Since $R(\mathfrak{g}) = 0$, we conclude that $(,)_ad |a$ is nondegenerate.

Therefore, $\mathfrak{g} = a \oplus a^\perp$, where $a$ is a minimal ideal, i.e. simple. (Note that $R(\mathfrak{g}) = 0$ so $a$ cannot be abelian.)

But now ideals of $a^\perp$ are ideals of $\mathfrak{g}$, so we can apply the same argument to $a^\perp$ since $R(\mathfrak{g}) = 0$ $\implies$ $R(a^\perp) = 0$. Therefore, $\mathfrak{g} = \oplus a_i$ where the $a_i$ are simple Lie algebras.

**Exercise 39.** Show that if $\mathfrak{g}$ is semisimple, then $\mathfrak{g}$ is the direct sum of minimal ideals in a unique manner. That is, if $\mathfrak{g} = \oplus a_i$, where $a_i$ are minimal, and $b$ is a minimal ideal of $\mathfrak{g}$, then in fact $b = a_i$ for some $i$. (Consider $b \cap a_j$.) Conclude that (1) $\implies$ (2).

**Lecture 8** Finally, we show that all derivations on semisimple Lie algebras are inner. Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $D : \mathfrak{g} \rightarrow \mathfrak{g}$ be a derivation. Cosider the linear functional $l : \mathfrak{g} \rightarrow k$ via $x \mapsto \text{trace } (D \cdot ad x)$. Since $\mathfrak{g}$ is semisimple, the Killing form is a nondegenerate inner product giving an isomorphism with the dual, so $\exists y \in \mathfrak{g}$ such that $l(x) = (y, x)_ad$. Our task is to show that $D = ad y$, i.e. that $E := D - ad y = 0$.

This is equivalent to showing that $Ex = 0$ for all $x \in \mathfrak{g}$, or equivalently that $(Ex, z)_{ad} = 0$ for all $z \in \mathfrak{g}$. Now,

$$\text{ad } (Ex) = E \cdot ad x - ad x \cdot E = [E, ad x] : \mathfrak{g} \rightarrow \mathfrak{g}$$

since $\text{ad } (Ex)(z) = [Ex, z] = E[x, z] - [x, Ez]$. Therefore,

$$(Ex, z)_{ad} = \text{trace } g(ad (Ex) \cdot ad z) = \text{trace } g([E, ad x] \cdot ad z) = \text{trace } g(E, [ad x, ad z]).$$
Exercise 40. (1) A nilpotent Lie algebra always has non-inner derivations.
(2) \( g = \langle a, b \rangle \) with \([a, b] = b\) only has inner derivations. Thus, this condition doesn’t characterise semisimplicity.

Exercise 41. (1) Let \( g \) be a simple Lie algebra, \( (,)_1 \) and \( (,)_2 \) two non-degenerate invariant bilinear forms. Show \( \exists \lambda \) such that \( (,)_1 = \lambda (,)_2 \).
(2) Let \( g = \mathfrak{sl}_n \mathbb{C} \). We will shortly show that this is a simple Lie algebra. We have two nondegenerate bilinear forms: the Killing form \((,)_\text{ad}\) and \( (A, B) = \text{trace} (AB) \) in the standard representation. Compute \( \lambda \).

5. Structure theory

Definition. A torus \( t \subseteq g \) is an abelian subalgebra s.t. \( \forall t \in t \) the adjoint \( \text{ad}(t) : g \to g \) is a diagonalisable linear map. A maximal torus is a torus that is not contained in any bigger torus.

Example 5.1. Let \( T = (S^1)^r \subseteq G \) where \( G \) is a compact Lie group (or \( T = (\mathbb{C}^*)^r \subseteq G \) where \( G \) is a reductive algebraic group). Then \( t = \text{Lie}(T) \subseteq g = \text{Lie}(G) \) is a torus, maximal if \( T \) is. (This comes from knowing that representations of \( t \) are diagonalisable by an averaging argument, but we don’t really know this.)

Exercise 42. (1) \( g = \mathfrak{sl}_n \) or \( g = \mathfrak{gl}_n \), \( t \) the diagonal matrices (of trace 0 if inside \( \mathfrak{sl}_n \)) is a maximal torus.
(2) \( \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \) is not a torus. (We should know this is not diagonalisable!)

For a vector space \( V \), let \( t_1, \ldots, t_r : V \to V \) be pairwise commuting diagonalisable linear maps; let \( \lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r \). Set \( V_\lambda = \{ v \in V : t_i v = \lambda_i v \} \) the simultaneous eigenspace.

Lemma 5.1. \( V = \bigoplus_{\lambda \in (\mathbb{C}^*)^r} V_\lambda \).

Proof. Induct on \( r \). For \( r = 1 \) follows from the requirement that \( t_1 \) be diagonalisable.

For \( r > 1 \), look at \( t_1, \ldots, t_{r-1} \) and decompose \( V = \bigoplus V_{(\lambda_1, \ldots, \lambda_{r-1})} \).

Since \( t_r \) commutes with \( t_1, \ldots, t_{r-1} \), it preserves the decomposition. Now decompose each \( V_{(\lambda_1, \ldots, \lambda_{r-1})} \) into eigenspaces for \( t_r \).

Let \( t \) be the \( r \)-dimensional abelian Lie algebra with basis \( (t_1, \ldots, t_r) \). The lemma asserts that \( V \) is a semisimple (i.e. completely reducible) representation of \( t \). \( V = \bigoplus V_\lambda \) is the decomposition into isotypic representations. Namely, letting \( \mathbb{C}_\lambda \) be the \( 1 \)-dimensional representation wherein \( t_i(w) = \lambda_i w \), we have \( \lambda \neq \mu \implies \mathbb{C}_\lambda \not\cong \mathbb{C}_\mu \), and \( V_\lambda \) is the direct sum of \( \dim V_\lambda \) copies of \( \mathbb{C}_\lambda \).

Exercise 43. Show that every irreducible representation of \( t \) is one-dimensional.

Another way of saying this is as follows. \( \lambda \) is a linear map \( t \to \mathbb{C} \), i.e. an element of the dual space \( \lambda \in t^* \) (sending \( \lambda(t_i) = \lambda_i \)). Therefore, one-dimensional representations of \( t \) (which are all the irreducible representations of \( t \)) correspond to elements of \( t^* = \text{Hom}_{\text{vector spaces}}(t, \mathbb{C}) \). The decomposition

\[ V = \bigoplus_{\lambda \in t^*} V_\lambda, \quad V_\lambda = \{ v \in V : t(v) = \lambda(t)v \} \]
is called the *weight space decomposition* of \( V \).

Now, let \( g \) be a Lie algebra and \( t \) a maximal torus. The weight space decomposition of \( g \) is

\[
g = g_0 \oplus \bigoplus_{\lambda \neq 0} g_\lambda
\]

where \( g_0 = \{ x \in g | [t, x] = 0 \} \) and \( g_\lambda = \{ x \in G : [t, x] = \lambda(t)x \} \).

**Definition.** \( R = \{ \lambda \in t^* | g_\lambda \neq 0, \lambda \neq 0 \} \) is the set of *roots* of \( g \).

We will now compute the root space decomposition for \( \mathfrak{sl}_n \).

**Example 5.2.** \( g = \mathfrak{sl}_n, t = \) diagonal matrices in \( \mathfrak{sl}_n \).

Let \( t = \begin{pmatrix} t_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_n \end{pmatrix} \) and \( E_{ij} \) the matrix with 1 in the \((i, j)\)th position and 0 elsewhere.

Then \([t, E_{ij}] = (t_i - t_j)E_{ij}\).

Let \( \epsilon_i \in t^* \), \( \epsilon_i(t) = t_i \). Then \( \epsilon_1, \ldots, \epsilon_n \) span \( t^* \), but \( \epsilon_1 + \cdots + \epsilon_n = 0 \).

**Remark.** \( t \) is a subalgebra of \( \mathbb{C}^n \), so \( t^* \) should be a quotient of \( (\mathbb{C}^n)^* \), and here is the relation by which it is a quotient.

Therefore, we can rewrite the above as \([t, E_{ij}] = (\epsilon_i - \epsilon_j)(t) \cdot E_{ij}\).

Therefore, the roots are \( R = \{ \epsilon_i - \epsilon_j, i \neq j \} \) and \( g_0 = t \). (Note that we’ve shown \( t \) is a maximal torus in the process.) The root space \( g_{\epsilon_i - \epsilon_j} = \mathbb{C}E_{ij} \) is one-dimensional.

That is, the root space decomposition of \( \mathfrak{sl}_n \) is:

\[
\mathfrak{sl}_n = t \oplus \bigoplus_{\epsilon_i - \epsilon_j \in R} g_{\epsilon_i - \epsilon_j}
\]

**Exercise 44.** “This is (a) an essential exercise, and (b) guaranteed to be on the exam. The first should get you to do it, the second is irrelevant.”

Compute the root space decomposition for \( \mathfrak{sl}_n \) (as we just did), \( \mathfrak{so}_{2n} \), \( \mathfrak{so}_{2n+1} \), \( \mathfrak{sp}_{2n} \) where \( t \) is the diagonal matrices in \( g \), and we take the conjugate \( \mathfrak{so}_n \) defined by

\[
\mathfrak{so}_n = \{ A \in \mathfrak{gl}_n | JA + A^TJ = 0 \}, \quad J = \begin{pmatrix} 0 & & & 1 \\ & \ddots & \ddots & \vdots \\ & & 0 & \cdots \\ 1 & \cdots & \cdots & 0 \end{pmatrix}
\]

(this is conjugate to the usual \( \mathfrak{so}_n \) since we’ve defined a nondegenerate inner product; in this version, the diagonal matrices are a maximal torus). The symplectic matrices are given by the same embedding as before.

In particular, show that \( t \) is a maximal torus and the root spaces are one-dimensional.

**Lecture 9** The Lie algebras we were working with last time are called the “classical” Lie algebras (they are automorphisms of vector spaces). The rest of the course will be concerned with explaining how their roots control their representations.

**Proposition 5.2.** \( \mathfrak{sl}_n \mathbb{C} \) is simple.
Proof. Recall

$$\mathfrak{sl}_n \mathbb{C} = t \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

where $R = \{\epsilon_i - \epsilon_j|i \neq j\}$ and $\mathfrak{g}_{\epsilon_i - \epsilon_j} = C E_{ij}$.

Suppose $r \subseteq \mathfrak{sl}_n$ is a nonzero ideal. Choose $r \in \mathfrak{r}$, $r \neq 0$, s.t. when we write $r = t + \sum_{\alpha \in R} e_\alpha$ with $e_\alpha \in \mathfrak{g}_\alpha$, the number of nonzero terms is minimal.

First, suppose $t \neq 0$. Choose $t_0 \in t$ such that $\alpha(t_0) \neq 0$ for all $\alpha \in R$ (i.e., $t_0$ has distinct eigenvalues). Consider $[t_0, r] = \sum_{\alpha \in R} \alpha(t_0)e_\alpha$. Note that this lies in $\mathfrak{r}$, and if it is nonzero, it has fewer nonzero terms than $r$, a contradiction. Thus, $[t_0, r] = 0$, i.e., $r = t \in t$.

Since $t \neq 0$, there exists $\alpha \in R$ with $\alpha(t) \neq 0$ (i.e. it can’t have the same eigenvalue, since it’s trace 0). Then

$$[t, e_\alpha] = \alpha(t)e_\alpha \implies e_\alpha \in \mathfrak{r}$$

Letting $\alpha = \epsilon_i - \epsilon_j$, we have $E_{ij} \in \mathfrak{r}$. Now,

$$[E_{ij}, E_{jk}] = E_{ik} \text{ for } i \neq k \quad [E_{si}, E_{ij}] = E_{sj} \text{ for } s \neq j$$

and therefore $E_{ab} \in \mathfrak{r}$ for $a \neq b$. Finally,

$$[E_{i,i+1}, E_{i+1,i}] = E_{ii} - E_{i+1,i+1} \in \mathfrak{r},$$

so in fact the entire basis for $\mathfrak{sl}_n$ lies in $\mathfrak{r}$ and $\mathfrak{r} = \mathfrak{sl}_n$.

Remark. This is actually a combinatorial statement about root spaces; we’ll see more about this later.

That leaves us with the case $t = 0$. If $r = cE_{ij}$ has only one nonzero term, we are done exactly as above; therefore,

$$r = e_\alpha + e_\beta + \sum_{\gamma \in R - \{\alpha, \beta\}} e_\gamma, \quad \alpha \neq \beta$$

Choose $t_0 \in \mathfrak{t}$ such that $\alpha(t_0) \neq \beta(t_0)$; then some linear combination of $[t_0, r]$ and $r$ will have fewer nonzero terms than $r$, a contradiction. \hfill \Box

Proposition 5.3. Let $\mathfrak{g}$ be a semisimple Lie algebra. Then maximal tori exist (i.e. are nonzero). Moreover, $\mathfrak{g}_0 = \{x \in \mathfrak{g}[t, x] = 0\} = t$. (Recall that $t$ is not defined to be the maximal abelian subalgebra, but as the maximal abelian subalgebra whose adjoint acts semisimply.)

We won’t prove this; the proof involves introducing the Cartan subalgebra and showing that it’s the same as a maximal torus. As a result, we will sometimes be calling $t$ the Cartan subalgebra.

This means that the root space decomposition of a semisimple Lie algebra is

$$\mathfrak{g} = t \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

as we have, of course, already seen for the classical Lie algebras.

Theorem 5.4 (Structure theorem for semisimple Lie algebras, part I). Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and let $t \subseteq \mathfrak{g}$ be a maximal torus. Write $\mathfrak{g} = t \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. Then

1. $CR = t^*$, i.e. the roots span $t^*$.
2. $\dim \mathfrak{g}_\alpha = 1$. 

If $\alpha, \beta \in R$ and $\alpha + \beta \in R$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha + \beta}$. If $\alpha + \beta \not\in R$ and $\alpha \neq -\beta$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$.

(4) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{t}$ is one-dimensional, and $\mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$. (In particular, $\alpha \in R \implies -\alpha \in R$.

**Exercise 45.** Check this for the classical Lie algebras.

**Proof.**

(1) Suppose not, then there exists $t \in \mathfrak{t}$ such that $\alpha(t) = 0$ for all $\alpha \in R$. But then $[t, \mathfrak{g}_\alpha] = 0$; as $[t, t] = 0$ by definition, we see that $t$ is central in $\mathfrak{g}$. However, $\mathfrak{g}$ is semisimple, so it has no abelian ideals, in particular no center.

We will now show a sequence of statements.

(a) $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda + \mu}$ if $\lambda, \mu \in \mathfrak{t}^*$.

Indeed, for $x \in \mathfrak{g}_\lambda$ and $y \in \mathfrak{g}_\mu$ we have

$$[t, [x, y]] = [[t, x], y] = [x, [t, y]] = (\lambda(t) + \mu(t))[x, y].$$

This shows that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\alpha + \beta}$ if $\alpha + \beta \in R$, is $0$ if $\alpha + \beta \not\in R$, and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{t}$.

(b) $(\mathfrak{g}_\lambda, \mathfrak{g}_\mu)_\text{ad} = 0$ if $\lambda \neq -\mu$; $(\mathfrak{g}_\lambda)_\text{ad} |_{\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}}$ is nondegenerate.

**Proof.** Let $x \in \mathfrak{g}_\lambda$ and $y \in \mathfrak{g}_\mu$. Recall $(x, y)_\text{ad} = \text{trace}_{\mathfrak{g}}(\text{ad} x \cdot \text{ad} y)$. To show that it’s zero, we show that $\text{ad} x \cdot \text{ad} y$ is nilpotent.

Indeed, $(\text{ad} x \cdot \text{ad} y)^N \mathfrak{g}_\alpha \subseteq \mathfrak{g}_{\alpha + N(\lambda + \mu)}$, so if $\lambda \neq -\mu$ then for $N \gg 0$ this is zero.

On the other hand, $(\mathfrak{g}_\lambda)_\text{ad}$ is nondegenerate and $\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{t}} \mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}$, so $(\mathfrak{g}_\lambda)_\text{ad} |_{\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}}$ must be nondegenerate.

(c) In particular, $(\mathfrak{g}_\lambda)_\text{ad} |_{\mathfrak{t}}$ is nondegenerate, as $\mathfrak{t} = \mathfrak{g}_0$.

**Remark.** $(\mathfrak{g}_\lambda)_\text{ad} |_{\mathfrak{t}} \neq (\mathfrak{g}_\lambda)_\text{ad} |_{\mathfrak{t}^*}$ (which is zero since $\mathfrak{t}$ is abelian).

Therefore, the Killing form defines an isomorphism $\nu : \mathfrak{t} \to \mathfrak{t}^*$ with $\nu(t)(t') = (t, t')_\text{ad}$. It also defines an induced inner product (well, symmetric bilinear form to be precise – we don’t know it’s positive) on $\mathfrak{t}^*$, via $(\nu(t), \nu(t')) = (t, t')_\text{ad}$.

(d) $\alpha \in R \implies -\alpha \in R$, since $(\mathfrak{g}_\alpha)_\text{ad}$ is nondegenerate on $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, but $(\mathfrak{g}_\alpha, \mathfrak{g}_\alpha)_\text{ad} = 0$ if $\alpha \neq 0$. In particular, $\mathfrak{g}_{-\alpha} \cong \mathfrak{g}_\alpha^*$ via the Killing form.

(e) $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha} \implies [x, y] = (x, y)_\text{ad} \nu^{-1}(\alpha)$.

**Proof.**

$$[t, [x, y]]_\text{ad} = ([t, x], y)_\text{ad} = \alpha(t)(x, y)_\text{ad} ,$$

which is exactly what we want. \(\square\)

(f) Let $e_\alpha \in \mathfrak{g}_\alpha$ be nonzero, and pick $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $(e_\alpha, e_{-\alpha})_\text{ad} \neq 0$. Then $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})_\text{ad} \nu^{-1}(\alpha)$. To show that $e_\alpha$, $e_{-\alpha}$, and their bracket give a copy of $\mathfrak{sl}_2$, we need to compute

$$[\nu^{-1}(\alpha), e_\alpha] = \alpha(\nu^{-1}(\alpha))e_\alpha = (\alpha, \alpha)e_\alpha.$$

We will be done if we show that $(\alpha, \alpha) \neq 0$ (then we get a copy of $\mathfrak{sl}_2$ after renormalising).

**Proposition 5.5.** $(\alpha, \alpha) \neq 0$ for all $\alpha \in R$.

**Proof.** Next time. \(\square\)
Lecture 10 We now prove the last proposition, \((\alpha, \alpha) \neq 0\) for all \(\alpha \in R\).

**Proof.** Suppose otherwise. Let \(m_\alpha = \langle e_\alpha, e_{-\alpha}, \nu^{-1}(\alpha) \rangle\). If \((\alpha, \alpha) = 0\), i.e. if \([\nu^{-1}(\alpha), e_\alpha] = 0\), then \([m_\alpha, m_\alpha] = \mathbb{C}\nu^{-1}\alpha\) and \(m_\alpha\) is solvable. However, by Lie’s theorem this implies that \(\text{ad} [m_\alpha, m_\alpha]\) acts by nilpotents on \(g\), i.e. \(\text{ad} \nu^{-1}(\alpha)\) is nilpotent. Since \(\nu^{-1}(\alpha) \in t\), it is also diagonalisable, and the only diagonalisable nilpotent element is 0. However, \(\alpha \in R \implies \alpha \neq 0\). \(\square\)

Now, define \(h_\alpha = \frac{2}{(\alpha, \alpha)}\nu^{-1}(\alpha)\) and rescale \(e_{-\alpha}\) such that \((e_\alpha, e_{-\alpha})_{\text{ad}} = 2/(\alpha, \alpha)\).

**Exercise 46.** The map \((e_\alpha, h_\alpha, e_{-\alpha}) \mapsto (e, h, f)\) gives an isomorphism \(m_\alpha \cong \mathfrak{sl}_2\).

**Remark.** In \(\mathfrak{sl}_n\), the root spaces are spanned by \(E_{ij}\), so we are saying that the subalgebra of matrices of the form

\[
\begin{pmatrix}
  a & * \\
  * & -a
\end{pmatrix}
\]

is a copy of \(\mathfrak{sl}_2\), which is obvious. The cool thing is that something like this happens for all the semisimple Lie algebras, and that this lets us know just about everything about them.

We will now use our knowledge of the representation theory of \(\mathfrak{sl}_2\). We show

**Claim.** \(\dim g_{-\alpha} = 1\) for all \(\alpha \in R\).

**Proof.** Pick a copy of \(m_\alpha = \langle e_\alpha, h_\alpha, e_{-\alpha} \rangle \cong \mathfrak{sl}_2\). Suppose \(\dim g_{-\alpha} > 1\), then the map \(g_{-\alpha} \to \mathbb{C}\nu^{-1}(\alpha), x \mapsto [e_\alpha, x]\) has a kernel. That is, there is a \(v \in g_{-\alpha}\) such that \(\text{ad} e_\alpha \cdot v = 0\). Then \(v\) is a highest-weight vector (it comes from a root space \(g_{-\alpha}\), so it’s an eigenvector of \(\text{ad} h_\alpha\)), and its weight is determined by

\[\text{ad} h_\alpha \cdot v = -\alpha(h_\alpha)v = -2v.\]

That is, \(v\) is a highest-weight vector of weight \(-2\). However, we know that in finite-dimensional representations of \(\mathfrak{sl}_2\) (which \(g\) is, since \(g_\alpha\) acts on it) the highest weights are nonnegative integers. This contradiction shows the claim. \(\square\)

Before we finish proving the theorem we had (we still need to show \([g_\alpha, g_\beta] = g_{\alpha + \beta}\) when \(\alpha, \beta, \alpha + \beta \in R\)), we show further combinatorial properties of the roots.

**Theorem 5.6** (Structure theorem, continued).

1. \[
\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z} \text{ for all } \alpha, \beta \in R
\]

2. If \(\alpha \in R\) and \(k\alpha \in R\), then \(k = \pm 1\).
The combinatorial statement of (3) is rather clunky. We now “unclunk” it:

\[ \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + \alpha k} \]

is an irreducible module for \((\mathfrak{sl}_2)_\alpha = \langle e_\alpha, h_\alpha, e_{-\alpha} \rangle\). In particular,

\[ \{k\alpha + \beta | k \in \mathbb{R}, k \in \mathbb{Z} \} \cup \{0\} \]

is of the form \(\beta - p\alpha, \beta - (p-1)\alpha, \ldots, \beta, \ldots, \beta + (q-1)\alpha, \beta + q\alpha\) where \(p - q = \frac{2(\alpha, \beta)}{\langle \alpha, \alpha \rangle}\). We call this set the “\(\alpha\) string through \(\beta\)”.

(1): let \(q = \max\{k : \beta + k\alpha \in \mathbb{R}\}\) and let \(v \in \mathfrak{g}_{\beta + q\alpha}\) be nonzero. Then \(\text{ad} e_\alpha \cdot v \in \mathfrak{g}_{\beta + (q+1)\alpha} = 0\), and

\[ \text{ad} h_\alpha \cdot v = (\beta + q\alpha)(h_\alpha) \cdot v = \left( \frac{2(\beta, \alpha)}{\langle \alpha, \alpha \rangle} + 2q \right) v. \]

That is, \(v\) is a highest weight vector for \((\mathfrak{sl}_2)_\alpha\) with weight \(2(\beta, \alpha)/(\alpha, \alpha) + 2q\). Since the weight in a finite-dimensional representation is an integer, we see that \(2(\beta, \alpha)/(\alpha, \alpha) \in \mathbb{Z}\).

Proof. (3): Structure of \(\mathfrak{sl}_2\)-modules tells us that \((\text{ad} e_{-\alpha})^r v \neq 0\) for \(0 \leq r \leq \frac{2(\alpha, \beta)}{\langle \alpha, \alpha \rangle} + 2q = N\), and that \((\text{ad} e_\alpha)^{N+1} v = 0\). Therefore, \(\{\beta + q\alpha - k\alpha, 0 \leq k \leq N\}\) are all roots (or possibly equal to zero – in any case, they have nonzero root space). This certainly is an irreducible \((\mathfrak{sl}_2)_\alpha\) module; we now need to show that there are no other roots in the \(\alpha\) string through \(\beta\).

We do this by repeating the same construction from the bottom up:

Let \(p = \max\{k : \beta - k\alpha \in \mathbb{R}\}\) and let \(w \in \mathfrak{g}_{\beta - p\alpha}\) be nonzero. Then \(\text{ad} e_{-\alpha} \cdot w = 0\) and \(\text{ad} h_\alpha \cdot w = (2(\beta, \alpha)/(\alpha, \alpha) - 2p)w\), so \(w\) is a lowest-weight vector of weight \(2(\beta, \alpha)/(\alpha, \alpha) - 2p\). By applying \(\text{ad} e_\alpha\) repeatedly, we get an irreducible \(\mathfrak{sl}_2\) module with roots \(\beta - p\alpha, \beta - (p-1)\alpha, \ldots, \beta\) and \(\langle 2(\beta, \alpha)/(\alpha, \alpha) \rangle\).

Now by construction we have \(p - 2(\alpha, \beta)/(\alpha, \alpha) \leq q\) and also \(q + 2(\alpha, \beta)/(\alpha, \alpha) \leq p\), which means that in fact \(p - q = 2(\alpha, \beta)/(\alpha, \alpha)\) and the two submodules we get coincide.

(2): By (1) we have \(2(\alpha, k\alpha)/(k\alpha, k\alpha) = 2/k \in \mathbb{Z}\) and \(2(k\alpha, \alpha)/(\alpha, \alpha) = 2k \in \mathbb{Z}\). Thus, it suffices to show that \(\alpha \in \mathbb{R} \implies 2\alpha \notin \mathbb{R}\).

But indeed, suppose \(2\alpha \in \mathbb{R}\) and let \(v \in \mathfrak{g}_{-2\alpha}\) be nonzero. Then \([e_\alpha, v] \in \mathfrak{g}_{-\alpha}\) has the property that

\[ ([e_\alpha, v], e_\alpha) = (v, [e_\alpha, e_\alpha]) = 0, \]

and since \(\mathfrak{g}_{\alpha}\) is one-dimensional and spanned by \(e_\alpha\), we find \(([e_\alpha, v], \mathfrak{g}_{\alpha}) = 0\). Since \((,)[\mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\alpha}]\) is nondegenerate, this means \([e_\alpha, v] = 0\), i.e. \(v \in \mathfrak{g}_{-2\alpha}\) is a highest-weight vector for \((\mathfrak{sl}_2)_\alpha\) with highest weight \(-4\). This doesn’t happen in finite-dimensional representations, so \(\mathfrak{g}_{-2\alpha} = 0\) and \(2\alpha\) is not a root.

(In fact, we could derive this from (3), since we know that there is a 3-dimensional \(\mathfrak{sl}_2\)-module running through 0.)

We finally show \([\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] = \mathfrak{g}_{\alpha + \beta}\) when \(\alpha, \beta, \alpha + \beta \in \mathbb{R}\). Indeed, we have just shown that \(\oplus \mathfrak{g}_{\beta + k\alpha}\) is an irreducible representation of \(\mathfrak{m}_\alpha\), and \(\text{ad} e_\alpha : \mathfrak{g}_{\beta + k\alpha} \to \mathfrak{g}_{\beta + (k+1)\alpha}\) is an isomorphism for \(k < q\). Since \(q \geq 1\), we see that \(\text{ad} e_\alpha \cdot \mathfrak{g}_{\beta} = \mathfrak{g}_{\alpha + \beta}\).

Remark. The combinatorial statement of (3) is rather clunky. We now “unclunk” it:

For \(\alpha \in \mathfrak{t}^*\), define \(s_\alpha : \mathfrak{t}^* \to \mathfrak{t}^*\) by \(s_\alpha(v) = v - \frac{2(\alpha, v)}{\langle \alpha, \alpha \rangle} \alpha\) (a reflection of \(v\) about \(\alpha\)).

Claim. (3) implies (and in fact, is equivalent to) \(s_\alpha(\beta) \in \mathbb{R}\) whenever \(\alpha, \beta \in \mathbb{R}\).
Proof. Let \( r = 2(\alpha, \beta)/\langle \alpha, \alpha \rangle \). If \( r \geq 0 \), then \( p = q + r \geq r \); if \( r \leq 0 \) then \( q = p - r \geq -r \). Either way, \( \beta - r\alpha \) is in the \( \alpha \) string through \( \beta \).

Over the next several lectures, we will classify the root systems as combinatorial objects. Before we do that, we state some further properties of the roots:

**Proposition 5.7.**

1. \( \alpha, \beta \in R \implies \langle \alpha, \beta \rangle \in \mathbb{Q} \).
2. If we pick a basis \( \beta_1, \ldots, \beta_l \) of \( t^* \), with \( \beta_i \in R \), then any \( \beta \in R \) expands as \( \sum q_i \beta_i \) with \( q_i \in \mathbb{Q} \), i.e. \( \dim \mathbb{Q} R = \dim \mathbb{C} t \).
3. \( \langle , \rangle \) is positive definite on \( \mathbb{Q} R \).

**Lecture 11**

**Proof of the Proposition at the end of last lecture.**

1. It suffices to show \( \langle \beta, \beta \rangle \in \mathbb{Q} \) for all \( \beta \in R \).

Let \( h, h' \in t \), then

\[
\langle h, h' \rangle_{\text{ad}} = \text{trace}_g(\text{ad} h \circ \text{ad} h') = \sum_{\alpha \in R} \alpha(h)\alpha(h')
\]

since \( g = t + \bigoplus_{\alpha \in R} g_\alpha \) by the structure theorem.

Therefore, for \( \lambda, \mu \in t^* \),

\[
(\lambda, \mu) = (\nu^{-1}(\lambda), \nu^{-1}(\mu))_{\text{ad}} = \sum_{\alpha \in R} \alpha(\nu^{-1}\lambda)\alpha(\nu^{-1}\mu) = \sum_{\alpha \in R} (\lambda, \alpha)(\mu, \alpha).
\]

In particular, \( \langle \beta, \beta \rangle = \sum_{\alpha \in R} \langle \alpha, \beta \rangle^2 \). Multiplying by \( 4/(\langle \beta, \beta \rangle)^2 \), we get

\[
\frac{4}{\langle \beta, \beta \rangle} = \sum_{\alpha \in R} \left( \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \right)^2 \in \mathbb{Z}
\]

and therefore \( \langle \beta, \beta \rangle \in \mathbb{Q} \).

2. Let \( B \) be the Gram matrix of the basis, \( B_{ij} = \langle \beta_i, \beta_j \rangle \).

**Exercise 47.** \( \langle , \rangle \) is a nondegenerate symmetric bilinear form \( \implies \det B \neq 0 \).

Let \( \beta = \sum c_i \beta_i \in R \). Then \( \langle \beta, \beta_i \rangle = \sum c_j \langle \beta_j, \beta_i \rangle \). Therefore, we can recover the vector of \( c_i \) as

\[
(c_1, \ldots, c_l) = ((\beta, \beta_1), \ldots, (\beta, \beta_l))(B^T)^{-1}
\]

Since the inner products of roots are all rational, we see that \( c_i \in \mathbb{Q} \) as well.

3. Let \( \lambda \in \mathbb{Q} R \). Then \( \lambda = \sum c_i \beta_i \) with \( c_i \in \mathbb{Q} \), and \( \langle \lambda, \alpha \rangle \in \mathbb{Q} \) for all \( \alpha \in R \). Moreover, \( \langle \lambda, \lambda \rangle = \sum_{\alpha \in R} \langle \alpha, \lambda \rangle^2 \geq 0 \), and if \( \langle \lambda, \lambda \rangle = 0 \) then \( \langle \lambda, \alpha \rangle = 0 \) for all \( \alpha \in R \). Since \( R \) spans \( t^* \) and \( \langle , \rangle \) is nondegenerate, this implies that \( \lambda = 0 \).

\[\square\]

6. **Root systems**

Let \( V \) be a vector space over \( \mathbb{R} \), \( \langle , \rangle \) an inner-product (i.e. a positive-definite symmetric bilinear form). For \( \alpha \in V \), \( \alpha \neq 0 \), write \( \alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle} \), so that \( \langle \alpha, \alpha^\vee \rangle = 2 \).

Define \( s_\alpha : V \to V \) by \( s_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha \).
Lemma 6.1. \( s_\alpha \) is the reflection in the hyperplane orthogonal to \( \alpha \). In particular, all but one of the eigenvalues of \( s_\alpha \) are 1, and the remaining one is \(-1\) (with eigenvector \( \alpha \)). Thus, \( s_\alpha^2 = 1 \), or \((s_\alpha + 1)(s_\alpha - 1) = 0\), and \( s_\alpha \in O(V, (,)) \), the orthogonal group of \( V \) defined by the inner product \((,\)).

Proof. Write \( V = \mathbb{R}\alpha \oplus \alpha^\perp \). Then \( s_\alpha(\alpha) = \alpha - (\alpha, \alpha^\vee)\alpha = -\alpha \), and \( s_\alpha \) fixes all \( v \in \alpha^\perp \). \( \square \)

Definition. A root system \( R \) in \( V \) is a finite set \( R \subseteq V \) such that

1. \( 0 \not\in R \), \( \mathbb{R}R = V \)
2. For all \( \alpha, \beta \in R \), the inner product \((\alpha, \beta^\vee) \in \mathbb{Z} \).
3. For all \( \alpha \in R \), \( s_\alpha(R) \subseteq R \). (In particular, \( s_\alpha(\alpha) = -\alpha \in R \).)
   Moreover, \( R \) is called reduced if
4. \( \alpha, k\alpha \in R \implies k = \pm 1 \).

Example 6.1. Let \( g \) be a semisimple Lie algebra, \( g = t + \bigoplus_{\alpha \in R} g_\alpha \). Then \( R \subseteq \mathbb{R}R \) is a reduced root system.

Definition. Let \( W \subseteq GL(V) \) be the group generated by reflections \( s_\alpha, \alpha \in R \); \( W \) is called the Weyl group of \( R \).

Observe that \( |W| < \infty \): \( W \) acts on \( R \) by permutations, and since \( 0 \not\in R \), the action is faithful (since \( s_\alpha \in W \) for all \( \alpha \in R \) and \( s_\alpha \) sends \( \alpha \) to \(-\alpha \), the only possible \( W \)-invariant element is 0, but it is not in \( R \)). Thus, \( W \) injects into \( Sym(R) \), so \( |R| < \infty \implies |W| < \infty \).

Definition. The rank of \( R \) is the dimension of \( V \).

Definition. An isomorphism of root systems \( (R, V) \rightarrow (R', V') \) is a linear bijection \( \phi : V \rightarrow V' \) such that \( \phi(R) = R' \). Note that \( \phi \) is not required to be an isometry.

Definition. If \( (R, V) \) and \( (R', V') \) are root systems, so is their direct sum \( (R \cup R', V \oplus V') \). A root system which is not isomorphic to a direct sum is called irreducible.

Example 6.2.

1. Rank 1: \( V = \mathbb{R} \), \((x, y) = xy \). \( R = \{\alpha, -\alpha\} \) with \( \alpha \neq 0 \). \( W = \mathbb{Z}/2 \).
   \(-\alpha \leftarrow \bullet \rightarrow \alpha \)

   Exercise 48. This is the only rank 1 root system.

2. Rank 2: \( V = \mathbb{R}^2 \) with the usual inner product.

   \( e_2 \)
   \( \leftarrow \rightarrow \)
   \( e_1 \)

   (a) This is \( A_1 \times A_1 \) (or \( A_1 + A_1 \)), and is not irreducible. The Weyl group is \( \mathbb{Z}/2 \times \mathbb{Z}/2 \).

   (b) \( A_2: \alpha = \alpha^\vee, \beta = \beta^\vee, (\alpha, \beta) = -1. \)
   \( W = S_3 \) is the dihedral group of order 6.
   This is the root system of \( \mathfrak{sl}_3 \), as you should be able to establish (and we will shortly see).
(c) $B_2$: $\alpha = e_1, \beta = e_2 - e_1$: $(\alpha, \alpha) = 1$ and $(\beta, \beta) = 2$. $\alpha$ and $\alpha + \beta$ are short roots; $\beta$ and $2\alpha + \beta$ are long roots.

$W$ is the dihedral group of order 8.

This is the root system of $\mathfrak{sp}_4$ and $\mathfrak{so}_5$.

(d) $G_2$:

At this point, we clearly should look at all the dihedral groups, right? Unfortunately, no: the condition $(\alpha, \beta)^\vee \in \mathbb{Z}$ rules out all but the dihedral group of order 12. This is the root system of a Lie group, but not one of the classical ones (symmetries of octonions).

**Exercise 49.** All of these are indeed root systems. They are the only rank 2 root systems, and $A_2, B_2,$ and $G_2$ are irreducible.

**Lemma 6.2.** If $R$ is a root system, so is $R^\vee = \{\alpha^\vee | \alpha \in R\}$.

**Exercise 50.** Prove it.

**Definition.** $R$ is *simply laced* if all the roots have the same length.

**Exercise 51.** If $R$ is simply laced, then $(R, V) \cong (R', V')$ where $(\alpha, \alpha) = 2$ for all $\alpha \in R'$ (i.e. $\alpha = \alpha^\vee$). [*Hint: If irreducible, rescale; if not, break up into a direct sum of irreducibles.*]

We would like to look at roots as coming from a lattice. We will see why length-2 is so useful.

**Definition.** A *lattice* $L$ is a finitely-generated free abelian group ($\cong \mathbb{Z}^l$ for some $l$) with bilinear form $L \oplus L \rightarrow \mathbb{Z}$, such that $(L \otimes \mathbb{R}, (,))$ is an inner product space.

A *root* of $L$ is $\alpha \in L$ such that $(\alpha, \alpha) = 2$. We write $R_L = \{l \in L | (l, l) = 2\}$ for the roots of $L$. 
Note that $\alpha \in R_L \implies s_\alpha(L) \subseteq L$.

**Lemma 6.3.** $R_L$ is a simply laced root system in $\mathbb{R}R_L$.

**Proof.** The only non-obvious claim is $|R_L| < \infty$. However, $R_L$ is the intersection of a compact set (sphere of radius $\sqrt{2}$) with a discrete set (the lattice), so is finite. □

**Definition.** We say that $L$ is generated by roots if $\mathbb{Z}R_L = L$. Note that then $L$ is an even lattice, i.e. $(l,l) \in 2\mathbb{Z}$ for all $l \in L$.

**Example 6.3.** $L = \mathbb{Z}\alpha$ with $(\alpha, \alpha) = \lambda$. If $\lambda = 2$ then $R_L = \{ \pm \alpha \}$ and $L = \mathbb{Z}R_L$; if $\frac{k^2\lambda}{2} \neq 1$ for all $k$ then $R_L = \emptyset$.

We now characterise the simply laced root systems.

1. **$A_n$.** Consider $\mathbb{Z}^{n+1} = \bigoplus_{i=1}^{n+1} \mathbb{Z}e_i$ with $(e_i, e_j) = \delta_{ij}$ (the square lattice). Define
   
   $L = \{ l \in \mathbb{Z}^{n+1} | (l, e_1 + \ldots + e_n) = 0 \} = \{ a_i e_i | \sum a_i = 0 \} \cong \mathbb{Z}^n$.

   Then $R_L = \{ e_i - e_j | i \neq j \}$, $\#R_L = n(n+1)$, $\mathbb{Z}R_L = L$.

   If $\alpha = e_i - e_j$ then $s_\alpha$ applied to a vector with coordinates $x_1, \ldots, x_{n+1}$ (in basis $e_1, \ldots, e_{n+1}$) swaps the $i$th and $j$th coordinate; therefore, $W = \langle s_{e_i-e_j} \rangle = S_{n+1}$.

   $(R_L, \mathbb{R}L)$ is the root system of type $A_n$. Note that $n$ is the rank of the root system, not the index of the associated Lie algebra.

2. **$D_n$.** Consider the square lattice $\mathbb{Z}^n$. The roots are $R_{\mathbb{Z}^n} = \{ \pm e_i \pm e_j | i \neq j \}$. Set
   
   $L = \mathbb{Z}R_L = \{ l = \sum a_i e_i | a_i \in \mathbb{Z}, \sum a_i \text{ even} \}$

   $s_{e_i-e_j}$ swaps the $i$th and $j$th coordinate as before; $s_{e_i+e_j}$ flips the sign of both the $i$th and the $j$th coordinate.

   $(R_L, \mathbb{Z}R_L)$ is called $D_n$; $\#R_L = 2n(n-1)$. The Weyl group is
   
   $W = (\mathbb{Z}/2)^{n-1} \rtimes S_n$

   where $(\mathbb{Z}/2)^{n-1}$ is the subgroup of even number of sign changes. (Possibly I mean $(\mathbb{Z}/2)^{n-1} \rtimes S_n$.)

**Exercise 52.**

(a) “You should not believe a word I say”: Check these statements!

(b) Draw $L \subseteq \mathbb{Z}^{n+1}$ and $R_L$ for $n = 1, 2$. Check that $A_1$ and $A_2$ agree with the previous pictures.

(c) Show that the root system of $\mathfrak{sl}_{n+1}$ has type $A_n$.

(d) The root system of $\mathfrak{so}_{2n}$ has type $D_n$.

**Exercise 53.**

(a) Check these statements!

(b) $D_n$ is irreducible if $n \geq 3$.

(c) $R_{D_3} = R_{A_3}$, $R_{D_4} = R_{A_1} \times R_{A_1}$

(d) The root system of $\mathfrak{so}_{2n}$ has type $D_n$.

**Lecture 12** We continue our classification of the simply laced root systems:

1. **$E_8$.** Let
   
   $\Gamma_n = \{ (k_1, \ldots, k_n) | \text{either all } k_i \in \mathbb{Z} \text{ or all } k_i \in \mathbb{Z} + \frac{1}{2} \text{ and } \sum k_i \in 2\mathbb{Z} \}$.

   Consider $\alpha = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in \Gamma_n$, and note that $(\alpha, \alpha) = n/4$. Thus, if $\Gamma_n$ is an even lattice, we must have $8|n$. 

Exercise 54. (a) $\Gamma_{8n}$ is a lattice.
(b) For $n > 1$ the roots of $\Gamma_{8n}$ are a root system of type $D_n$
(c) The roots in $\Gamma_8$ are \{±$e_i$, ±$e_j$, $i \neq j$; $\frac{1}{2}$($±e_1$, ±$e_2$, ±$e_3$), with an even number of $-$ signs\}.

The root system of $\Gamma_8$ is called the root system of type $E_8$. The number of roots is
$$\#R_{\Gamma_8} = \binom{8}{2} \cdot 4 + 128 = 240.$$ 
To check that you’re awake, the dimension of the associated Lie algebra is $280 + \text{rank}(R_{\Gamma_8}) = 248$.

Remark. $E_8$ is weird. Its smallest representation is the adjoint representation of dimension 248. This is quite unlike $so_n$, $sl_n$, etc., which had dimension $O(n^2)$ and an $n$-dimensional representation. There isn’t a good understanding of why $E_8$ is like that.

Exercise 55. Can you compute $\#W$, the order of the Weyl group of $E_8$? [Answer: $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$]

Exercise 56. If $R$ is a root system and $\alpha \in R$, then $\alpha^\perp \cap R$ is a root system.

We will now apply this to $\Gamma_8$. Let $\alpha = \frac{1}{2}(1, 1, \ldots, 1)$ and $\beta = e_7 + e_8$.

(2)

Definition. $\alpha^\perp \cap R_{\Gamma_8}$ is a root system of type $E_7$.
$\langle \alpha, \beta \rangle^\perp \cap R_{\Gamma_8}$ is a root system of type $E_6$.

Exercise 57. Show $\#R_{E_7} = 126$ and $\#R_{E_6} = 72$. Describe the corresponding root lattices.

Remark. These lattices are reasonably natural objects. Indeed, if we took the associated algebraic group and its maximal torus, there is a natural lattice associated with it coming from the fundamental group. Equivalently, we can think of the group homomorphisms to (or from) $\mathbb{C}^\times$. This isn’t quite the root lattice, but it’s closely related (and we might say more about it later).

Theorem 6.4. (1) ("ADE classification") The complete list of irreducible simply laced root systems is

$$A_n, n \geq 1, \quad D_n, n \geq 4, \quad E_6, \quad E_7, \quad E_8.$$ 

No two root systems in this list are isomorphic.

(2) The remaining irreducible reduced root systems are denoted by

$$B_2 = C_2, \quad B_n, C_n, n \geq 3, \quad F_4, \quad G_2$$

where

$$R_{B_n} = \{\pm e_i, \pm e_i \pm e_j, i \neq j\} \subseteq \mathbb{Z}^n$$
$$R_{C_n} = \{\pm 2e_i, \pm e_i \pm e_j, i \neq j\} \subseteq \mathbb{Z}^n$$

with $R_{C_n}^\perp = R_{B_n}$. The root system of type $B_n$ corresponds to $so_{2n+1}$, and the root system of type $C_n$ corresponds to $sp_{2n}$. The Weyl groups are

$$W_{B_n} = W_{C_n} = (\mathbb{Z}/2)^n \rtimes S_n.$$
The root system $F_4$ is defined as follows: let $Q = \{(k_1, \ldots, k_4) | \text{all } k_i \in \mathbb{Z} \text{ or all } k_i \in \mathbb{Z} + \frac{1}{2}\}$ and take

$$R_{F_4} = \{ \alpha \in Q | (\alpha, \alpha) = 2 \text{ or } (\alpha, \alpha) = 1 \} = \{ \pm e_i, \pm e_i \pm e_j, i \neq j, \frac{1}{2}(\pm e_1 \pm \ldots \pm e_4) \}.$$ 

**Remark.** The duality between $R_{B_n}$ and $R_{C_n}$ suggests some form of duality between $\mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$. We will see it later.

To prove this theorem, we will first choose a good basis for $V$ (the space spanned by the root lattice). Choose $f : V \rightarrow \mathbb{R}$ linear such that $f(\alpha) \neq 0$ for all $\alpha \in R$.

**Definition.** A root $\alpha \in R$ is positive, $\alpha \in R^+$, if $f(\alpha) > 0$.

A root $\alpha \in R$ is negative, $\alpha \in R^-$, if $f(\alpha) < 0$. (Equivalently, $-\alpha \in R^+$.)

A root $\alpha \in R^+$ is simple, $\alpha \in \Pi$, if $\alpha$ is not the sum of two positive roots.

**Example 6.4.** (1) $A_n$: the roots are $R = \{ e_i - e_j | i \neq j \}$. Choose $f(e_1) = n + 1, f(e_2) = n, \ldots, f(e_{n+1}) = 1$.

Then $R^+ = \{ e_i - e_j | i < j \}$.

Since $f(R) \in \mathbb{Z}$, if $f(\alpha) = 1$ then $\alpha$ must be simple. Therefore, $\Pi \supseteq \{ e_1 - e_2, \ldots, e_n - e_{n+1} \}$. On the other hand, it is clear that the positive integer span of these is $R^+$, so in fact we have equality.

(2) $B_n$: $R = \{ \pm e_i, \pm e_i \pm e_j, i \neq j \}$. Take $f(e_1) = n, \ldots, f(e_n) = 1$. Then $R^+ = \{ e_i, e_i \pm e_j | i < j \}$ and $\Pi = \{ e_1 - e_2, \ldots, e_{n-1} - e_n, e_n \}$.

(3) $C_n$: $R = \{ \pm 2e_i, \pm e_i \pm e_j, i \neq j \}$. $R^+ = \{ 2e_i, e_i \pm e_j, i < j \}$ and $\Pi = \{ e_1 - e_2, \ldots, e_{n-1} - e_n \}$.

(4) $D_n$: $R = \{ \pm e_i \pm e_j, i \neq j \}$, $\Pi = \{ e_1 - e_2, \ldots, e_{n-1} - e_n, e_{n-1} + e_n \}$.

(5) $E_8$: set $f(e_1) = 28 = 1 + 2 + \ldots + 7$, and $f(e_i) = 9 - i$ for $i = 2 \ldots 8$. Then $R^+ = \{ e_i \pm e_j, i < j, \frac{1}{2}(e_1 \pm e_2 \pm \ldots \pm e_8) \text{ with an even number of } - \text{ signs} \}$

and

$$\Pi = \{ e_2 - e_3, \ldots, e_7 - e_8, \frac{1}{2}(e_1 + e_8 - e_2 - \ldots - e_7), e_7 + e_8 \}_{f=1}^{f=3}$$

**Exercise 58.** Check all of these (by finding a suitable function $f$ where one is not listed).

Also find the simple roots for $E_6, E_7, F_4$, and $G_2$.

**Remark.** So far we have made two choices. We have chosen a maximal torus, and we have chosen a function $f$. Neither of these matters. We may or may not prove this about the maximal torus later; as for $f$, note that the space of functionals $f$ is partitioned into cells within which we get the same notions of positive roots. The Weyl group acts transitively on these cells, which should show that the choice of $f$ does not affect the structure of the root system we are deriving.

**Proposition 6.5.** (1) $\alpha, \beta \in \Pi \implies \alpha - \beta \notin R$.

(2) $\alpha, \beta \in \Pi, \alpha \neq \beta \implies (\alpha, \beta') \leq 0$ (i.e. the angle between $\alpha$ and $\beta$ is obtuse).

(3) Every $\alpha \in R^+$ can be written as $\alpha = \sum k_i \alpha_i$ with $\alpha_i \in \Pi$ and $k_i \in \mathbb{Z}_{\geq 0}$.

(4) Simple roots are linearly independent.

(5) $\alpha \in R^+, \alpha \notin \Pi \implies \exists \alpha_i \in \Pi$ such that $\alpha - \alpha_i \in R^+$.
Exercise 59. Prove this! (Either by a case-by-case analysis – which you should do in a few cases anyway – or by finding a uniform proof from the axioms of a root system.)

We will now find another way to represent the root system. Let \( \Pi = \{\alpha_1, \ldots, \alpha_l\} \), let

\[ a_{ij} = (\alpha_i, \alpha_j). \]

**Definition.** \( A = (a_{ij}) \) is the Cartan matrix.

**Properties 6.6.**

1. \( a_{ij} \in \mathbb{Z}, a_{ii} = 2, a_{ij} \leq 0 \) for \( i \neq j \)
2. \( a_{ij} = 0 \iff a_{ji} = 0 \)
3. \( \det A > 0 \)
4. All principal subminors of \( A \) have positive determinant.

**Proof.** The first two should be obvious from the above. To prove the determinant, note that

\[ A = \begin{pmatrix}
  2 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & 2
\end{pmatrix} \cdot \left( \begin{pmatrix}
  (\alpha_1, \alpha_1) \\
  \vdots \\
  (\alpha_l, \alpha_l)
\end{pmatrix} \right). \]

The first matrix is diagonal with positive entries, the second one is the Gram matrix of a positive definite bilinear form and so has positive determinant.

The last property follows because any principal subminor of \( A \) (i.e. obtained by removing the same set of rows and columns) has the same form as \( A \).

We will represent \( A \) by a Dynkin diagram. The vertices correspond to simple roots. Between vertices \( \alpha_i \) and \( \alpha_j \) there are \( a_{ij} \) lines (note that this value can be 0 or 1 in the simply laced diagrams, may be 2 in \( B_n \) or \( C_n \), and may be 3 in \( G_2 \)). If there are 2 or 3 lines, we put an arrow towards the short root.

**Exercise 60.** Show the Dynkin diagrams are as we claim.

**Lecture 13**

**Exercise 61.** If \((R, V)\) is an irreducible root system with positive roots \( R^+ \), then there is a unique root \( \theta \in R^+ \) such that \( \forall \alpha \in \Pi, \theta + \alpha \not\in R^+ \). This is called the highest root.

At the moment, the easiest way to prove this is to examine all the root systems we have. Later we'll give a uniform proof (and show the relationship between \( \theta \) and the adjoint representation).

**Example 6.5.** In \( A_n \), the highest root is \( e_1 - e_{n+1} \).

Define the “extended Cartan matrix” \( \tilde{A} \) by setting \( \alpha_0 = -\theta \) (and then \( \tilde{A} = (a_{ij})_{0 \leq i,j \leq l} \) with \( a_{ij} = (\alpha_i, \alpha_j) = \frac{2(a_{ij})}{(\alpha_i, \alpha_i)} \)).

The associated diagram is called the “extended” or “affine” Dynkin diagram.

**Example 6.6.** \( A_1 \) has \( A = (2) \) and \( \tilde{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \).
\[ A_n \]

\[ D_n \]

\[ E_8 \]

\[ E_7 \]

\[ E_6 \]

\[ B_n \]

\[ C_n \]

\[ G_2 \]

\[ F_4 \]

\textbf{Figure 2.} Dynkin diagrams (simply laced first).

\[ A_n \text{ has} \]

\[
A = \begin{pmatrix}
2 & -1 & & \\
-1 & 2 & \ddots & \\
& \ddots & \ddots & -1 \\
& & -1 & 2
\end{pmatrix}
\quad \quad
\tilde{A} = \begin{pmatrix}
2 & -1 & 0 & -1 \\
-1 & 2 & -1 & \\
0 & -1 & 2 & \ddots & 0 \\
\vdots & & \ddots & \ddots & -1 \\
-1 & 0 & -1 & 2
\end{pmatrix}
\]

Notice that \( \tilde{A} \) satisfies almost the same conditions as the Cartan matrix, i.e.

(1) \( a_{ij} \in \mathbb{Z}, a_{ij} \leq 0 \) for \( i \neq j \), \( a_{ii} = 2 \)

(2) \( a_{ij} = 0 \iff a_{ji} = 0 \)
(3) \( \det \tilde{A} = 0 \), and every principal subminor of \( \tilde{A} \) has positive determinant.  

The third property follows since \( \tilde{A} \) is related to the Gram matrix for \( \alpha_0, \ldots, \alpha_l \), which are not linearly independent.

**Exercise 62.** Write down \( \tilde{A} \) and the extended Dynkin diagram for all types. To get you started, in \( B_n \) we have \( \alpha_0 = -e_1 - e_2 \), in \( C_n \) we have \( \alpha_0 = -2e_1 \), and in \( D_n \) we have \( \alpha_0 = -e_1 - e_2 \).

**Exercise 63.** Show that \( \tilde{A}_n^{(2)} \) (“twisted \( A_n \)” ) also has determinant 0. (See Figure 3.)

**Exercise 64.** The Dynkin diagram of \( \tilde{A}^T \) is the Dynkin diagram of \( A \) with the arrows reversed.

These diagrams (plus their transposes) are (almost) all the non-garbage Lie algebras out there. (We’ll be slightly more precise about the meaning of “non-garbage” later.)

**Theorem 6.7.** An irreducible (i.e., connected) Dynkin diagram is one of \( A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, \) and \( G_2 \).

**Proof.**

1. First, we classify the rank-2 Dynkin diagrams, i.e. their Cartan matrices:
   \[
   A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}
   \]
   with \( \det A = 4 - ab > 0 \). This leaves the options \( (a, b) = (0, 0) \) \( (A_1 \times A_1) \), \( (1, 1) \) \( (A_2) \), \( (2, 1) \) or \( (1, 2) \) \( (B_2) \), and \( (3, 1) \) or \( (1, 3) \) \( (G_2) \).

2. Observe any subdiagram of a Dynkin diagram is a Dynkin diagram. In particular, since the affine Dynkin diagrams have determinant 0, they are not subdiagrams of a Dynkin diagram.

3. A Dynkin diagram contains no cycles. Indeed, let \( \alpha_1, \ldots, \alpha_n \) be distinct simple roots, and consider \( \alpha = \sum \alpha_i/\sqrt{\langle \alpha_i, \alpha_i \rangle} \). Then
   \[
   0 < \langle \alpha, \alpha \rangle = n + \sum_{i < j} \frac{2\langle \alpha_i, \alpha_j \rangle}{\sqrt{\langle \alpha_i, \alpha_i \rangle \langle \alpha_j, \alpha_j \rangle}} = n - \sum_{i < j} \sqrt{a_{ij}a_{ji}}.
   \]
   Therefore, \( \sum_{i < j} \sqrt{a_{ij}a_{ji}} < n \). On the other hand, if we had a cycle on \( n \) simple roots, it would have at least \( n \) edges, giving \( \sum_{i < j} \sqrt{a_{ij}a_{ji}} \geq n \), a contradiction.

4. If a diagram is simply laced, it is \( A, D, \) or \( E \):
   Suppose it’s not of type \( A \). Since \( \tilde{D}_4 \) is not a subdiagram, any branchings are at most 3-way; and since \( \tilde{D}_n \) is not a subdiagram for \( n > 4 \), there is exactly one branch point. Therefore, the diagram is T-shaped, with \( p, q, \) and \( r \) vertices in each leg. We call this shape \( T_{p,q,r} \); e.g., \( E_8 = T_{5,3,2} \).

**Exercise 65.** Finish the proof in one of two ways:
   (a) By using the fact that \( \tilde{E}_6, \tilde{E}_7, \tilde{E}_8 \) are not subdiagrams of a Dynkin diagram; or
   (b) By showing \( \det T_{p,q,r} = pq + pr + qr - pqr \), and concluding that \( D_n, E_6, E_7, \) and \( E_8 \) are the only possibilities. (E.g.: \( \det E_8 = 15 + 10 + 6 - 30 = 1 \).)

5. We now deal with the non-simply-laced diagrams.

**Exercise 66.** If there is a triple bond in the diagram, then the Dynkin diagram is \( G_2 \). (Note that we took care of one of the possibilities in \( G_2 \); you need to check the possibility of a double and a triple bond, and of two triple bonds.)
Finally, if there is a double bond in the diagram, then by $\tilde{C}_n$ and $\tilde{A}_n^{(2)}$ there’s only one of them, and the diagram does not branch by $\tilde{B}_n$. If the double bond is in the middle, we have $F_4$ (since we don’t have $\tilde{F}_4$); otherwise, we have $B_n$ or $C_n$. 

\[ \square \]
Exercise 6.7. Compute the determinant of all the Cartan matrices.

Example 6.7.

\[ A_1 : \det(2) = 2 \quad A_2 : \det \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3 \quad A_3 : \det = 4 \ldots \]

This comes out of the following: \( SL_{n+1} = \{ X : \det X = 1 \} \) has center consisting of \((n+1)\)th roots of unity (i.e. a cyclic group of order \(n + 1\)). Note that the order of the center is equal to the determinant of the Cartan matrix. This is true in general: the order of the center of a simply connected algebraic group with Lie algebra \( \mathfrak{g} \) and associated Cartan matrix \( A \) is equal to \( \det A \).

In particular, this means that the algebraic group attached to \( E_8 \) has no center.

7. Existence and uniqueness

Lecture 14

A. Independence of choices

Theorem 7.1. All maximal tori are conjugate. (Recall that we are in the setting of \( \text{char } k = 0, k = \overline{k} \).) That is, if \( t \) and \( t' \) are two maximal tori of \( \mathfrak{g} \),

\[ \exists g \in \text{Aut}(\mathfrak{g})^0 = \{ g \in GL(\mathfrak{g}) | g : \mathfrak{g} \to \mathfrak{g} \text{ is a Lie algebra homomorphism} \}^0 \]

such that \( gt = t' \). (Note: \( \text{Aut}(\mathfrak{g}) \) is itself an algebraic group; \( \text{Aut}(\mathfrak{g})^0 \) is the connected component of 1. Also, \( \text{Lie}(\text{Aut}\mathfrak{g}) = \mathfrak{g} \).)

The proof is not very hard and involves a little bit of algebraic geometry; we won’t give it.

Theorem 7.2. All positive root systems \( R^+ \) are conjugate. That is, let \( (V, R) \) be a root system. Let \( f_1, f_2 : V \to \mathbb{R} \) satisfy \( f_i(\alpha) \neq 0 \) for all \( \alpha \in \mathbb{R} \), and define \( R^+_1, E^+_2 \). There exists a unique \( w \in W \) such that \( wR^+_1 = R^+_2 \). (Consequently, \( w\Pi_1 = \Pi_2 \) and we get the same Cartan matrix out of the two choices.)

The proof is quite easy, but we still won’t give it.

Corollary 7.3. \( \mathfrak{g} \) determines the Cartan matrix, i.e. the Dynkin diagram, regardless of the choices we made.

B. Uniqueness

Theorem 7.4. For \( i = 1, 2 \), let \( \mathfrak{g}_i \) be semisimple Lie algebras with Cartan subalgebras \( t_i \), root systems \( R_i \), positive root systems \( R^+_i \), simple roots \( \Pi_i \) and Cartan matrices \( A_i \). Suppose that, after reordering indices, \( A_1 = A_2 \). Then there exists an isomorphism of Lie algebras \( \phi : \mathfrak{g}_1 \to \mathfrak{g}_2 \) such that \( \phi(t_1) = t_2 \), \( \phi(R_1) = R_2 \), etc.

We will not prove this theorem, but we will say more about it.

C. Existence

Theorem 7.5. Let \( A \) be a Cartan matrix. There exists a semisimple Lie algebra whose Cartan matrix is \( A \).

Remark. We already know this for the infinite families, but a nice proof would not be case-by-case.
We will now talk about existence and uniqueness.

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{g} = \mathfrak{t} + \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, and let the simple roots be $\Pi = \{\alpha_1, \ldots, \alpha_l\}$. Choose

$$0 \neq E_i \in \mathfrak{g}_{\alpha_i}, \ F_i \in \mathfrak{g}_{-\alpha_i} \text{ such that } (E_i, F_i)_{ad} = \frac{2}{(\alpha_i, \alpha_i)}, \ H_i = \frac{2\nu^{-1}(\alpha_i)}{(\alpha_i, \alpha_i)}.$$  

Define $a_{ij} = (\alpha_i^\vee, \alpha_j) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$.

We have the following commutation relations:

- $[H_i, H_j] = 0$ (the torus commutes)
- $[H_i, E_j] = a_{ij}E_j$  
  $([H_i, E_j] = \alpha_j(H_i)E_j = \alpha_j(2\nu^{-1}(\alpha_i))/(\alpha_i, \alpha_i)E_j = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)E_j)$
- $[H_i, F_j] = -a_{ij}F_j$
- $[E_i, F_j] = 0$ for $i \neq j$, since $[E_i, F_j] \in \mathfrak{g}_{\alpha_i, -\alpha_j}$ and $\alpha_i - \alpha_j \not\in R$ for $\alpha_i, \alpha_j \in \Pi$
- $[E_i, F_i] = H_i$

"Depending on your mood, these are either some of the relations or all of the relations". We will see what this means.

Let $\mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha, \ \mathfrak{n}^- = \bigoplus_{\alpha \in -R^+} \mathfrak{g}_\alpha$. Then $\mathfrak{g} = \mathfrak{n}^+ + \mathfrak{t} + \mathfrak{n}^-$. (Note that $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are not ideals of $\mathfrak{g}$.)

**Lemma 7.6.** $E_i$ generate $\mathfrak{n}^+$; $F_i$ generate $\mathfrak{n}^-$; hence $E_i, F_i$ generate $\mathfrak{g}$ as Lie algebras.

**Proof.** For a root $\alpha \in R^+$ write $\alpha = \sum k_i \alpha_i$ with $k_i \in \mathbb{Z}_{\geq 0}$, and define the height of $\alpha$ to be $h(\alpha) = \sum k_i$. We now induct on the height of the root. Recall that if $\alpha \in R^+$ and $\alpha$ is not simple, then there exists $\alpha_i \in \Pi$ such that $\alpha - \alpha_i = \beta \in R^+$. Therefore, $\mathfrak{g}_\beta$ is generated by the $E_i$. However, then $\mathfrak{g}_\alpha = [\mathfrak{g}_\beta, \mathfrak{g}_{\alpha_i}]$ is also generated by the $E_i$.

The corresponding claim about $F_i$ follows immediately by taking $-R^+$ as the positive roots.

Finally, $[E_i, F_i] = H_i$ and $\{H_i| i = 1, \ldots, l\}$ are a basis for $\mathfrak{t}$ (recall the simple roots are a basis for $\mathfrak{t}^*$).

Let $A$ be a "generalised Cartan matrix". That is, $a_{ii} = 2$, $a_{ij} = 0$ whenever $a_{ji} = 0$, and $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$ (but we do not include requirements of positive definiteness). Let

$$\tilde{\mathfrak{g}} = \langle E_i, F_i, H_i| [H_i, H_j] = 0, [H_i, E_j] = a_{ij}E_j, [H_i, F_j] = -a_{ij}F_j, [E_i, F_j] = 0 \text{ for } i \neq j, [E_i, F_i] = H_i \rangle$$

Clearly, if $A$ is the Cartan matrix of $\mathfrak{g}$, then $\tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$. Are these all the relations? Not quite.

Let

$$\overline{\mathfrak{g}} = \tilde{\mathfrak{g}}/\langle (\text{ad } E_i)^{1-a_{ij}} E_j = 0, (\text{ad } F_j)^{1-a_{ij}} F_j = 0 \rangle$$

(e.g., if $a_{ij} = 0$ then $E_i$ and $E_j$ commute, and if $a_{ij} = -1$ then $[E_i, [E_i, E_j]] = 0$. These are called the "Serre relations."

**Remark.** They are called the Serre relations because they were discovered by Harish-Chandra and Chevalley.

**Exercise 68.** Check that the Serre relations hold in the classical groups.

**Theorem 7.7.** (1) If $A$ is indecomposable, $\tilde{\mathfrak{g}}$ has a unique maximal ideal, and $\overline{\mathfrak{g}}$ is the quotient of $\tilde{\mathfrak{g}}$ by its maximal ideal, so is simple. (But not necessarily finite-dimensional!)
Hence, if $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra with Cartan matrix $A$, then the map $\tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ via $E_i \mapsto E_i$, $F_i \mapsto F_i$, $H_i \mapsto H_i$ (where on the left-hand side we have the canonical generators of $\tilde{\mathfrak{g}}$ and on the right hand side we have the elements of the root spaces we found above) factors through $\tilde{\mathfrak{g}}$ and is surjective, so gives an isomorphism $\tilde{\mathfrak{g}} \cong \mathfrak{g}$.

The second part of the theorem clearly implies the uniqueness. We also see that existence is equivalent to the statement

$\tilde{\mathfrak{g}}$ is finite-dimensional $\Leftrightarrow$ $A$ is a Cartan matrix

**Definition.** $\overline{\mathfrak{g}}$ (for any $\mathfrak{A}$) is called a Kac-Moody algebra (this time because they were independently discovered by Kac and Moody).

Most of the representation theory we will be talking about holds verbatim for the Kac-Moody algebras. However, note that all we have given for them is a finite presentation, which is not a very nice object. In particular, the root spaces of these algebras are not 1-dimensional. The dimension of the root space for any given root is computable but horrible, unless the matrix $A$ is an affine Dynkin diagram. In that case, the Lie algebra is approximately $\mathbb{C}c + \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$.

The Weyl group has a corresponding presentation:

**Theorem 7.8 (Presentation of the Weyl group).** Write $s_i = s_{\alpha_i}$, then

$W = \langle s_1, \ldots, s_l | s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle,$

where $m_{ij}$ are defined in terms of the Cartan matrix as follows:

\[
\begin{array}{c|cccc}
  & a_{ij} & a_{ji} & 0 & 1 & 2 & 3 \\
  m_{ij} & 2 & 3 & 4 & 6
\end{array}
\]

E.g., if $i$ and $j$ are not connected in the Dynkin diagram then $s_i s_j = s_j s_i$, and if they are connected by a single edge, then $s_i s_j s_i = s_j s_i s_j$. These are known as the “braid relations”. In $A_{n-1}$ they look like this:

\[
\begin{pmatrix}
  & & \\
  & & \\
\end{pmatrix}
= \begin{pmatrix}
  & & \\
  & & \\
\end{pmatrix}
\]

**Figure 4.** Braid relations in $A_{n-1}$. We are asserting that the permutations are equal.

**Lecture 15**

**Exercise 69.** Check, for each root system, that the relations claimed for the Weyl group do hold. (This requires checking it for the rank-2 root systems.) We will not prove the isomorphism.

**Remark.** To show existence, we know that $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, so in principle existence should only require picking a basis $E_\alpha$ and computing $[E_\alpha, E_\beta] = c_{\alpha\beta} E_{\alpha + \beta}$. How hard is it to write down $c_{\alpha\beta}$?
For a long time it was thought to be quite hard; however, with the discovery of the affine Lie algebras, it turned out to be quite easy. We might even get to doing it for the ADE algebras.

**Remark.** So far the status of existence of a finite-dimensional simple Lie algebra corresponding to an abstract root system for us is as fikkiws: we have shown that there exists a simple Lie algebra corresponding to each generalised Cartan matrix, but not that it is finite-dimensional. Suppose that for the $g$ that we have defined we write $g = t + \bigoplus_{\alpha \in R} g_\alpha$ (where the torus comes from the root system). We can ask two questions: (1) what is the relationship between $R$ and the root system we started with?; (2) What is the dimension of $\dim g_\alpha$?

While it is true that $W \cdot \Pi \subseteq R$, in general there is not an equality between them. We call the roots in $W \cdot \Pi$ “real roots” $R^r$; for the real roots, $\dim g_\alpha = 1$. For finite-dimensional Lie algebras, all roots are real. If $A$ is an affine Cartan matrix, then the semidefinite nature of it means that there exists a unique (up to scaling) root of length 0, called the imaginary root. The imaginary roots are pleasant and describable objects. For a general, indefinite, Cartan matrix, there exists an algorithm for computing $\dim g_\alpha$, but the numbers are not very meaningful.

8. **Representations**

Let $g$ be a semisimple Lie algebra, $g = t + \bigoplus_{\alpha \in R} g_\alpha = t + n^+ + n^-$. Let $V$ be a finite-dimensional representation of $g$. (We don’t quite need finite-dimensional; we’ll see exactly what we need a bit later.)

**Proposition 8.1.**

1. $V = \bigoplus_{\lambda \in \mathbb{L}} V_\lambda$, where $V_\lambda = \{v \in V | t(v) = \lambda(t)v \ \forall t \in t\}$ (this is the weight space decomposition with respect to $t$)

2. If $V_\lambda \neq 0$, then $\lambda(h_\alpha) \in \mathbb{Z}$ for all $\alpha \in R$. (Here, $h_\alpha$ is the element of the $(\mathfrak{sl}_2)_\alpha$, $h_\alpha = \nu^{-1}(\alpha^\vee)$.)

**Proof.** Since $V$ is a finite-dimensional representation of $g$, it is a finite-dimensional representation of $(\mathfrak{sl}_2)_\alpha$. Therefore, $h_\alpha$ acts diagonisably on $V$ with $\lambda(h_\alpha) \in \mathbb{Z}$. The first assertion follows because $h_\alpha$ span $t$. \hfill \Box

**Definition.** $Q = ZR = \bigoplus_{\alpha \in \Pi} Z\alpha_i$ is the lattice of roots.

$P = \{\gamma \in Q | (\gamma, \alpha^\vee) \in Z \ \forall \alpha \in R\}$ (the dual lattice) is the lattice of weights. Note that the condition can be checked by computing $(\gamma, \alpha_i^\vee)$ for $\alpha_i \in \Pi$.

Since, for $\beta, \alpha \in R$ we have $(\beta, \alpha^\vee) \in \mathbb{Z}$, we see $Q \subseteq P$. Also, if $V$ is a finite-dimensional representation of $g$ and $V_\lambda \neq 0$, then $\lambda \in P$, as $\lambda(h_\alpha) = (\lambda, \alpha^\vee)$.

**Exercise 70.**

1. $|P/Q| < \infty$. In fact, $|P/Q| = \det A$ where $A$ is the Cartan matrix.

2. $W(P) \subseteq P$, where $W$ is the Weyl group.

**Example 8.1.** In $\mathfrak{sl}_2$, $R = \pm \alpha$, $Q = \mathbb{Z}\alpha$, $(\alpha, \alpha) = 2$, so $P = \mathbb{Z}\alpha/2$, and $|P/Q| = 2 = \det(2)$.

**Definition.** For $V$ a finite-dimensional representation of $g$, define

$$\text{ch} V = \sum_{\lambda \in P} \dim V_\lambda \cdot e^\lambda \in \mathbb{Z}[P].$$

Here, $e^\lambda$ is a formal symbol $(e^\lambda e^\mu = e^{\lambda+\mu})$ forming a basis for $\mathbb{Z}[P]$.

“People are looking at me as if I’m speaking some language I don’t speak, let alone you don’t speak.”
Example 8.2. For $\mathfrak{sl}_2$, $P = \mathbb{Z} \alpha/2$. Write $z = e^{\alpha/2}$, then $\text{ch} L(n) = z^n + \ldots + z^{-n} = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}}$.  

Example 8.3. Let $\mathfrak{g} = \mathfrak{sl}_3$, and $V = \mathfrak{g}$ the adjoint representation. Then 

$$\text{ch} V = 2 + z + w + zw + z^{-1} + w^{-1} + z^{-1}w^{-1},$$

where $w = e^{\alpha_1}$, $z = e^{\alpha_2}$.

![Figure 5. Root lattice for $\mathfrak{sl}_3$, with dimensions of weight spaces. Note that the outer numbers are 1, and the picture is $W$-invariant ($W = S_3$ here).](image)

Proposition 8.2. If $V$ is a finite-dimensional representation of $\mathfrak{g}$, then $\dim V_\lambda = \dim V_{w\lambda}$ for all $w \in W$, i.e. $\text{ch} V$ is $W$-invariant, $\text{ch} V \in \mathbb{Z}[P]^W$. (In particular, for $\mathfrak{sl}_n$ this means that $\text{ch} V$ are symmetric functions.)

We will sketch three proofs, of which the third one is the easiest, but the first two reveal what is actually going on.

Sketch 1. If $G$ is an algebraic group with Lie algebra $\mathfrak{g}$ then (by a theorem) $W = N(T)/T$ (where $T$ is the maximal torus in $G$ whose Lie algebra is $\mathfrak{t}$, and $N(T)$ is its normaliser).

Example 8.4. If $G = \text{SL}_n$ with $T$ the diagonal matrices, then $N(T)$ are matrices with only one nonzero entry in each row and column, and $N(T)/T = S_n$ the permutation matrices.

That is, $\exists \bar{w} \in N(T)$ such that $\bar{w}T = w$.

If $G$ acts on $V$ (as it always does when $G$ is simply connected), then $\bar{w}(V_\lambda) = V_{w\lambda}$, since $t\bar{w}(v) = \bar{w}(t^{-1}t\bar{w})v = \bar{w}(\lambda(t^{-1}t\bar{w}))v$.

That is, $W$ is almost a subgroup of $G$. However, it is definitely not actually a subgroup. E.g., in $\text{SL}_2$, we take $\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} = \bar{s}$, and then $s^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in T$ is not equal to 1. That is, $W$ does not embed into $G$ and does not act on $V$. (There is a small 2-group, $\approx (\mathbb{Z}/2)^l$, that intervenes. This 2-group is also involved in determining the coefficients $c_{\alpha\beta}$.)

Sketch 2. We can mimic the above argument in $\mathfrak{g}$.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \exp(f) \exp(-e) \exp(f).$$

Therefore, for each root $\alpha$ we define $s_\alpha = \exp(f_\alpha) \exp(-e_\alpha) \exp(f_\alpha)$, where the matrix exponential is defined in the usual power series way.
Exercise 71. (1) If $V$ is a finite-dimensional representation of $\mathfrak{g}$, then $e_\alpha, f_\alpha$ act nilpotently on $\mathfrak{g}$, so $s_\alpha : V \to V$ is a well-defined finite sum.
(2) $s_\alpha^2 = e_\alpha : V_\lambda \to V_\lambda$, where $e_\alpha^2 = 1$ and $e_\alpha$ acts by a constant. Determine this constant (in terms of $\lambda$).
(3) $s_\alpha(V_\lambda) = V_{\lambda - m\alpha}$.

Remark. We did not really need $V$ to be finite-dimensional; what we need instead is for each $e_\alpha$ and $f_\alpha$ to act locally nilpotently. We say $x : V \to V$ acts locally nilpotently if for all $v \in V$ we have $x^n v = 0$ for some $n$.

Exercise 72. This is equivalent to saying that as an $\mathfrak{sl}_2$-module, $V$ splits into a (possibly infinite) direct sum of finite-dimensional $\mathfrak{sl}_2$-modules.

Definition. Such a $V$ is called integrable (because you can “integrate” the Lie algebra action to a group action).

In the rest of the course, you can replace all instances of “finite-dimensional” by “integrable”, and then all theorems and proofs will work for the Kac-Moody algebras as well.

Sketch 3. If we’re just working with $\mathfrak{sl}_2$ anyway, we shouldn’t need all the other stuff.

Exercise 73. Consider $V$ as a representation of $(\mathfrak{sl}_2)_\alpha \times T$. Then $V$ splits up into a direct sum of strings, each of which is of the form
$$
\lambda \quad \lambda - \alpha \quad \lambda - 2\alpha \quad \ldots \quad \lambda - m\alpha
$$
where $m = \lambda(h_\alpha)$.

Such a string is obviously $s_\alpha$-invariant, since $s_\alpha$ will just flip the string from left to right (as the reflection did in $\mathfrak{sl}_2$).

Lecture 16

Definition. For $\lambda, \mu \in T^*$ write $\mu \leq \lambda$ to mean $\lambda - \mu = \sum k_i \alpha_i$ with $k_i \in \mathbb{N}$. The points $\{\mu \in P | \mu \leq \lambda\}$ are the lattice points in an obtuse cone.

![Figure 6. Cone $\leq \lambda$ for $\mathfrak{sl}_3$.](image)

Definition. If $V$ is a representation of $\mathfrak{g}$, we say that $\mu \in P$ is a highest weight if $V_\mu \neq 0$ and $V_\lambda \neq 0 \implies \lambda \leq \mu$.

We say $v \in V_\gamma$ is a singular vector if $v \neq 0$ and $e_\alpha v = 0$ for all $\alpha \in R^+$. (Note that $e_\alpha v \in V_{\gamma + \alpha}$, so if $\gamma$ is a highest weight, then all nonzero elements of $V_\gamma$ are singular vectors.)

We say that $\mu$ is an extremal weight if $w\mu$ is a highest weight for some $w \in W$.

Example 8.5. In the adjoint rep of $\mathfrak{sl}_3$, the highest weight is $\alpha_1 + \alpha_2$, and the entire outer hexagon are extremal weights.
Theorem 8.3. \( \mathfrak{g} \) is a semisimple Lie algebra over \( \mathbb{C} \).

1. "Complete reducibility": if \( V \) is a finite-dimensional representation of \( \mathfrak{g} \), then \( V \) is a direct sum of irreducible representations of \( \mathfrak{g} \).

Set \( P^+ = \{ \lambda \in P : (\lambda, \alpha^\vee) > 0 \forall \alpha \in \mathcal{R}^+ \} \) (of course, it’s enough to check this on \( \alpha \in \Pi \)). This is the cone of dominant weights. The assertion of the next parts of the theorem is that \( P^+ \) parametrises the finite-dimensional irreducible representations of \( \mathfrak{g} \), with \( \lambda \mapsto L(\lambda) \). More precisely,

2. Let \( V \) be a finite-dimensional irreducible representation of \( \mathfrak{g} \), and \( v \in V_\lambda \) a singular vector. Then

   - \( \text{dim} V_\lambda = 1 \)
   - \( \lambda \neq 0 \implies \mu \leq \lambda \), so \( \lambda \) is a highest weight (and \( v \) a highest weight vector)
   - Moreover, if \( W \) is another finite-dimensional irreducible representation of \( \mathfrak{g} \) with highest weight \( \lambda \), and \( w \in W_\lambda \), \( w \neq 0 \), then there exists a unique isomorphism \( V \rightarrow W \) mapping \( v \mapsto w \). (Of course, given they are irreducible representations, if there is an isomorphism, there is a one-dimensional family of them, so we would pin down the exact one by sending \( v \mapsto w \). That is, uniqueness is not the nontrivial statement here.)

3. Given \( \lambda \in P^+ \), there exists a finite-dimensional irreducible representation with highest weight \( \lambda \), called \( L(\lambda) \).

4. A formula for \( \text{ch} L(\lambda) \) (the Weyl character formula), to be stated later.

Remark. For Kac-Moody algebras, we need a slight modification, since it is no longer obvious that we will have highest-weight vectors. We need to work in the category of integrable highest-weight representations instead.

Corollary 8.4.

\[
\text{ch} L(\lambda) = e^\lambda + \sum_{\mu < \lambda} a_{\mu \lambda} e^\mu \in \mathbb{Z}[P]
\]

hence \( \{ \text{ch} L(\lambda) | \lambda \in P^+ \} \) are linearly independent (they have different leading terms). On the other hand,

\[
\text{ch} L(\lambda) = m_\lambda + \sum_{\mu < \lambda} \tilde{a}_{\mu \lambda} m_\mu \in \mathbb{Z}[P]^W, \text{ where } m_\mu = \sum_{\gamma \in W_\mu} e^\gamma.
\]

Since \( \{ m_\mu \} \) clearly form a basis for \( \mathbb{Z}[P]^W \), \( \text{ch} L(\lambda) \) are a basis of \( \mathbb{Z}[P]^W \).

Corollary 8.5. If \( V \) and \( W \) are finite-dimensional representations of \( \mathfrak{g} \), \( V \cong W \Leftrightarrow \text{ch} V = \text{ch} W \).

Proof. Complete reducibility + characters of irreducibles are a basis. \( \square \)

Definition. Define \( \omega_i \in P \) to be the dual basis to simple (co)roots, \( (\omega_i, \alpha_j^\vee) = \delta_{ij} \) for \( i = 1, \ldots, l \). These are called the fundamental weights.

Then \( P^+ = \oplus \mathbb{Z}_{\geq 0} \omega_i = \{ \sum n_i \omega_i | n_i \geq 0 \} \). This is a fundamental domain for the action of \( W \). Each irreducible representation will have a unique highest weight lying in this cone, which is what parametrises the representation.

Exercise 74. Compute \( \omega_i \) for the simple Lie algebras \( \mathfrak{sl}_n, \mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n} \), etc., and draw the picture for \( A_2, B_2, \) and \( G_2 \).
For any \( \lambda \in P \) \( \implies \lambda = \sum \lambda(h_i)\omega_i \), where \( h_i = \mu^{-1}(\alpha^\vee_i) \).

**Example 8.6.** For any \( g \), the trivial representation is \( L(0) \).

For any simple \( g \), the adjoint representation \( V = g \) is irreducible. The decomposition \( g = t + \bigoplus_{\alpha \in R} g_\alpha \) means that a highest weight \( \lambda \) is a root such that \( \lambda + \alpha_i \notin R \) for all \( i \), i.e. \( \lambda = \theta \), the highest root in \( R^+ \). (Therefore, the theorem of the highest weight implies that \( \theta \) is unique, as promised.)

**Example 8.7.** In \( A_{n-1} \), the highest root is \( \theta = e_1 - e_n \) (recall \( \alpha_i = e_i - e_{i+1} \) and \( h_i \) is a diagonal matrix with \(+1\) in the \( i \)th position, \(-1\) in the \((i+1)\)th position, and \(0\) elsewhere).

We have \( \theta(h_1) = 1, \theta(h_2) = 0, \ldots, \theta(h_{n-2}) = 0, \theta(h_{n-1}) = 1 \). We can label the Dynkin diagram by \( \theta(h_i) \):

\[
\begin{array}{cccc}
1 & 0 & \ldots & 0 & 1 \\
\end{array}
\]

**Figure 8.** Adjoint representation of \( \mathfrak{sl}_n \); labels are \( \theta(h_i) \), giving a decomposition of \( \theta \) into fundamental weights.

**Exercise 76.** Compute \( \theta(h_i) \) for all the simple Lie algebras.

**Example 8.8.** Other representations of \( \mathfrak{sl}_n \): we have the standard representation, \( \mathbb{C}^n \), with weight basis \( v_1, \ldots, v_n \) and weights \( e_1, \ldots, e_n \) (recall \( \sum e_i = 0 \)). The highest weight is \( e_1 \) (as \( e_1 > e_2 > \ldots > e_n \), since \( e_1 - e_2, e_2 - e_3, \ldots \in R^+ \)). Therefore, \( \mathbb{C}^n = L(\omega_1) \), and \( \text{ch} \mathbb{C}^n = e^{e_1} + \ldots + e^{e_n} = z_1 + \ldots + z_n \) where \( z_i = \exp(e_i) \in \mathbb{Z}[P] = \mathbb{Z}[z_1^\pm 1, \ldots, z_n^\pm 1]/(z_1 \ldots z_n = 1) \).

Now, if \( V \) and \( W \) are representations of \( g \), then so is \( V \otimes W \) (via \( x \in g \) acts by \( x \otimes 1 + 1 \otimes x \)). In particular, \( V \otimes V \) is a representation. Note that \( \sigma : V \otimes V \to V \otimes V, a \otimes b \mapsto b \otimes a \), commutes with the \( g \)-action, so eigenspaces of \( \sigma \) are \( g \)-modules. That is, \( S^2V \) and \( \Lambda^2V \) are \( g \)-modules. In general, these may not be irreducible, but they are for \( \mathfrak{sl}_n \) (provided \( V \) is irreducible, of course!), as we check below:

Consider \( \Lambda^s \mathbb{C}^n \) (for \( s \leq n - 1 \)). This has weight basis \( v_{i_1} \wedge \ldots \wedge v_{i_s} \) for \( i_1 < \ldots < i_s \), and the weight of this is \( e_{i_1} + \ldots + e_{i_s} \) as is clear from the action. Therefore,

\[
E_i(v_{i_1} \wedge \ldots \wedge v_{i_s}) = 0 \text{ for all } i \iff v_{i_1} \wedge \ldots \wedge v_{i_s} = v_1 \wedge \ldots \wedge v_s,
\]
Let \( V \) be any algebra with a non-degenerate invariant bilinear form \((.,.)\), e.g., \( \mathfrak{g} \) semisimple and \( (.,.) = (.,.)_{ad} \). Let \( x_1, \ldots, x_N \) be a basis of \( \mathfrak{g} \), and \( x^1, \ldots, x^N \) the dual basis (so \( (x_i, x^j) = \delta_{ij} \)).

**Definition.** \( \Omega = \sum x_i x^i \) is the *Casimir of \( \mathfrak{g} \).*

**Remark.** For the moment, \( \Omega \) is an operator on any representation of \( \mathfrak{g} \). However, we will in a moment define the "universal enveloping algebra of \( \mathfrak{g} \)", and that is where \( \Omega \) naturally lives.

**Lemma 8.6.** For all \( x \in \mathfrak{g} \), we have [\( \Omega, x \)] = 0.

**Proof 1, coordinate-based.** [\( \Omega, x \)] = [\( \sum x_i x^i, x \)]. Since \( \text{ad} \, x \) is a derivation, we rewrite this as \( \sum x_i [x^i, x] + \sum [x_i, x] x^i \).

Write \([x^i, x] = \sum a_{ij} x^j\) and \([x_i, x] = \sum b_{ij} x^j\). Then
\[
a_{ij} = (\sum [x^i, x_j], x) = (\sum x_j [x^i, x], x) \\
b_{ij} = ([x_i, x^j], x) = ([x^j, x_i], x) = -a_{ji}
\]
Thus,
\[ [\Omega, x] = \sum x_i x^i a_{ij} + \sum x_j x^j b_{ij} = 0. \]

Proof 2, without coordinates. We have maps of $g$-modules $C \hookrightarrow \text{End}(g) \cong g \otimes g^* \cong g \otimes g$ (the first map is via $\lambda \mapsto \lambda \text{Id}$, i.e. $1 \mapsto \sum x_i \otimes x^i$; the last isomorphism is via the bilinear form). Since $g$ acts on $V$, we have a map of $g$-modules $g \rightarrow \text{End}(V)$. Finally, we have a map of $g$-modules $\text{End}(V) \otimes \text{End}(V) \rightarrow \text{End}(V)$ by multiplication. This finally gives
\[ C \hookrightarrow g \otimes g^* \cong g \otimes g \rightarrow \text{End}(V) \otimes \text{End}(V) \rightarrow \text{End}(V) \]

sending $1 \mapsto \Omega$. Therefore, $\Omega$ generates a trivial $g$-submodule of $\text{End}(V)$, which is equivalent to saying $[\Omega, x] = 0$ for all $x \in g$.

Now let $g$ be semisimple, $g = t \oplus \bigoplus_{\alpha \in R} g_{\alpha}$, and $(\cdot, \cdot)_{\text{ad}}$ the Killing form. Choose a basis $u_1, \ldots, u_l$ of $t$, $x_\alpha$ of $g_\alpha$; choose the dual basis $u^1, \ldots, u^l$ of $t$ and $x^\alpha$ of $g_{\alpha}$ (so that $(x_\alpha, x^\alpha) = 1$). Normalise $x_\alpha$ so that $(x_\alpha, x^\alpha)_{\text{ad}} = 1$, so that $x^\alpha = x_{\alpha}$. Then $[x_\alpha, x_{\alpha}] = \nu^{-1}(\alpha)$ by definition. Now,
\[ \Omega = \sum u_i u^i + \sum_{\alpha \in R^+} (x_\alpha x_{\alpha} + x_{\alpha} x_\alpha) = \sum u_i u^i + 2 \sum_{\alpha \in R^+} x_{\alpha} x_\alpha + \sum_{\alpha \in R^+} \nu^{-1}(\alpha). \]

Definition. $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$.

Remark. Of course, there will be an exercise to compute $\rho$ for all the simple Lie algebras. However, we will state this exercise later, when it’s more obvious what $\rho$ is.

Remark. If you like algebraic geometry, this is the square root of the canonical bundle on the flag variety.

Then
\[ \Omega = \sum u_i u^i + 2\nu^{-1}(\rho) + 2 \sum_{\alpha \in R^+} x_{\alpha} x^\alpha. \]

Lemma 8.7. Let $V$ be a $g$-module, $v \in V$ a singular vector with weight $\lambda$ (i.e. $n^+ v = 0$ and $tv = \lambda(t)v$). Then $\Omega v = (|\lambda + \rho|^2 - |\rho|^2)v$.

Proof. Apply the most recent version of $\Omega$:
\[ \Omega v = \left( \sum \lambda(u_i) \lambda(u^i) + \lambda(2\nu^{-1}(\rho)) \right) v + 2 \sum_{\alpha \in R^+} x_{\alpha} x^\alpha v = ((\lambda, \lambda) + 2(\lambda, \rho))v. \]

In particular, if $V$ is irreducible, $\Omega$ acts on $V$ by $(\lambda, \lambda) + 2(\lambda, \rho)$ by Schur’s lemma.

9. PBW theorem

Let $g$ be any Lie algebra over a field $k$.

Definition. The universal enveloping algebra of $g \mathcal{U}g$ is the associative algebra over $k$ generated by $g$ with relations $xy - yx = [x, y]$ for all $x, y \in g$. 
More formally, if $V$ is a vector space over $k$, then
$$TV = k + V + V \otimes V + V^\otimes 3 + \ldots = \bigoplus_{n \geq 0} V^\otimes n$$
is the tensor algebra of $V$, and is the free associative algebra generated by $V$. Let $J$ be the two-sided ideal in $Tg$ generated by $x \otimes y - y \otimes x = [x, y]$ for all $x, y \in g$; then $Ug = Tg/J$.

**Exercise 80.** An enveloping algebra for $g$ is a linear map $i : g \to A$, where $A$ is an associative algebra, and $i(x)i(y) - i(y)i(x) = i([x, y])$. (For example, if $V$ is a representation of $g$, we can take $A = \text{End}(V)$ and $i$ the action map.) Show that $Ug$ is initial in the category of enveloping algebras, i.e. any such map $i$ factors uniquely through $Ug$.

**Remark.** This is old terminology – anywhere other than the above exercise and old textbooks, “enveloping algebra” almost certainly means “universal enveloping algebra”.

The Casimir operator naturally lives in $Ug$; in fact, as we’ve shown, in the center $Z(Ug)$.

Observe that $Tg$ is a graded algebra, but that the relations we are imposing on it are not homogeneous ($x \otimes y$ and $y \otimes x$ have degree 2, while $[x, y]$ has degree 1). Consequently, $Ug$ is not graded. It is, however, filtered.

Define $(Ug)_n$ to be the span of products of $\leq n$ elements of $g$, then $U_n \subseteq U_m$ for $n \leq m$, and $U_n \cdot U_m \subseteq U_{n+m}$. (In particular, $U_0 \supseteq k$, $U_1 \supseteq k + g$, etc.)

**Exercise 81.** Show that if $x \in U_m$ and $y \in U_n$, then $[x, y] \in U_{n+m-1}$ (we clearly have $[x, y] \in U_{n+m}$, and are asserting that in fact the degree is one lower). By abuse of notation, for $x, y \in Ug$ we write $[x, y]$ to mean $xy - yx$.

**Definition.** For a filtration $F_0 \subseteq F_1 \subseteq \ldots$, we write $gr(F) = \oplus F_i/F_{i-1}$. (We probably need $F_i$ to be a filtration of an algebra for this to make sense...) This is the associated graded algebra.

**Theorem 9.1** (Poincaré-Birkhoff-Witt).

$$grUg = \oplus (Ug)_n/((Ug)_{n-1}) \cong Sg$$

Equivalently, if $\{x_1, \ldots, x_n\}$ is a basis for $g$, then $\{x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} | a_i \in \mathbb{Z}_{\geq 0}\}$ is a basis for $Ug$. In particular, $g \hookrightarrow Ug$.

**Exercise 82.**

1. Show that the previous exercise, $[U_n, U_m] \subseteq U_{n+m-1}$, implies that there is a well-defined map $S(g) \to grUg$, extending the map $g \mapsto g$.

2. Show that the map is surjective, or equivalently that $\{\prod x_i^{a_i}\}$ span $Ug$.

Therefore, the hard part of PBW theorem is injectivity. We will not prove the theorem.

**Remark.** The theorem should imply that $Ug$ is a deformation of $Sg$ (i.e. if we consider $xy - yx = t[x, y]$ then for $t = 0$ we get $Sg$ whereas for $t = 1$ – and, in fact, for generic $t$ by renormalising – we get $Ug$). The shortest proof of the theorem is indeed via deformation theory.

**10. Back to representations**

**Exercise 83.** If $V$ is a representation of $g$ and $v \in V$, then the $g$-submodule of $V$ generated by $v$ is $Ug \cdot v$.

**Lecture 18**
**Definition.** $V$ is a highest weight module for $\mathfrak{g}$ if $\exists v \in V$ singular (i.e. $n^+v = 0$ and $t \cdot v = \lambda(t)v$ for all $t \in \mathfrak{t}$ and some $\lambda \in \mathfrak{t}^*$) such that $V = U\mathfrak{g} \cdot v$.

Note that in fact $\mathfrak{u}n^- \cdot v = V$.

**Proof.** PBW theorem (the easy direction) implies that if $x_1, \ldots, x_N$ is a basis of $\mathfrak{g}$ then $x_1^{a_1} \cdots x_N^{a_N}$ spans $U\mathfrak{g}$. Taking a basis for $\mathfrak{g}$ ordered “$\mathfrak{n}^-$, then $\mathfrak{t}$, then $\mathfrak{n}^+$”, we see $U\mathfrak{g} \cong \mathfrak{u}n^- \otimes U\mathfrak{t} \otimes \mathfrak{u}n^+$, and $\mathfrak{u}n^+ \cdot v = U\mathfrak{t} \cdot v = \mathbb{C} v \subseteq \mathfrak{u}n^- \cdot v$ (since $\mathfrak{n}^+ \cdot v = 0$).

**Remark.** If $V$ is irreducible and finite-dimensional, it is a highest weight module (as it has a singular vector $v$, and $U\mathfrak{g} \cdot v$ is a submodule).

**Proposition 10.1.** Let $V$ be a highest weight module for $\mathfrak{g}$ (not necessarily finite-dimensional), and let $v_\Lambda$ be a highest weight vector with highest weight $\Lambda \in \mathfrak{t}^*$.

1. $\mathfrak{t}$ acts diagonalisably on $V$, so $V = \bigoplus_{\mu \in D(\Lambda)} V_\mu$, where

\[ D(\Lambda) = \{ \Lambda - \sum k_i \alpha_i | k_i \in \mathbb{Z}_{\geq 0} \} = \{ \mu \in \mathfrak{t}^* | \mu \leq \Lambda \} \]

(the “descent” of $\Lambda$).

2. $V_\Lambda = \mathbb{C} v_\Lambda$, and all other weight spaces are finite-dimensional.

3. $V$ is irreducible if and only if all singular vectors are in $V_\Lambda$.

4. $\Omega$ acts on $V$ as $|\Lambda + \rho|^2 - |\rho|^2$

5. If $v_\lambda$ is a singular vector, then $|\lambda + \rho|^2 = |\Lambda + \rho|^2$

6. If $\Lambda \in \mathbb{R}R$, then there exist only finitely many $\lambda$ such that $V_\lambda$ contains a singular vector.

7. $V$ contains a unique maximal proper submodule $I$. $I$ is graded by $\mathfrak{t}$, $I = \bigoplus (I \cap V_\lambda)$, and $I$ is the sum of all proper submodules of $V$.

**Remark.** The assumption in (6) is in fact unnecessary, but there isn’t quite a one-line proof in that case. Note that if $V$ is finite-dimensional, then we in fact know that all weights lie in $\mathbb{Q}R$, but we are not assuming finite dimensionality here.

**Proof.** As $V = \mathfrak{u}n^- \cdot v_\Lambda$, expressions $e_{-\beta_1} e_{-\beta_2} \cdots e_{-\beta_r} v_\Lambda$ span $V$ (where $\beta_i \in R^+$ and $e_{-\beta_i} \in \mathfrak{g}_{-\beta_i}$). The weight of such an expression is $\Lambda - \beta_1 - \beta_2 - \cdots - \beta_r$.

**Exercise 84.** Prove it!

This implies (1) immediately, and (2) upon noticing that there are finitely many ways of writing $\lambda = \Lambda - \beta_1 - \cdots - \beta_r$ for $\beta_i \in R^+$.

3: if $v_\lambda \in V_\lambda$ is a singular vector, then $N = U\mathfrak{g} \cdot v_\lambda = \mathfrak{u}n^- \cdot v_\lambda$ is a submodule of $V$ with weights in $D(\lambda)$. Since $\lambda \neq \Lambda$, we have $D(\lambda) \subsetneq D(\Lambda)$, whence $N$ is a proper submodule. Thus, if there is a singular vector outside $V_\lambda$, then $V$ is not irreducible.

Conversely, if $N \subseteq V$ is a proper submodule, then $\mathfrak{t}N \subseteq N$, so $N$ is graded by $\mathfrak{t}$, and its weights lie in $D(\Lambda)$. Let $\lambda = \Lambda - \sum k_i \alpha_i$ be a weight in $N$, with $\alpha_i \in \Pi$ and $\sum k_i$ minimal. Since $N$ is a proper submodule, $\lambda \neq \Lambda$ so $\sum k_i > 0$, and in that case any nonzero vector in $N_\lambda$ is singular (since $\lambda + \alpha_i$ is not a weight of $N$ for any simple $\alpha_i$).

7: Any proper submodule of $V$ is $\mathfrak{t}$-graded and does not contain $v_\lambda$. The sum of all proper submodules still does not contain $v_\lambda$, so it is the maximal proper submodule.

4: for the singular vector $v_\Lambda$ we have $\Omega v_\Lambda = (|\Lambda + \rho|^2 - |\rho|^2)v_\Lambda$. Since $V = U\mathfrak{g} \cdot v_\Lambda$ and $\Omega$ commutes with $U\mathfrak{g}$, we see that $\Omega$ acts by the same constant on all of $V$.

5: is now immediate (apply the same observation to any other singular vector).
(6): if \( V_\lambda \) contains a singular vector, then \(|\lambda + \rho|^2 = |\Lambda + \rho|^2\). This means that \( \lambda \) lies in the intersection of a sphere in \( \mathbb{R}^R \) (compact) and \( D(\Lambda) \) (discrete), so there are only finitely many such \( \lambda \). \( \square \)

**Definition.** Let \( \Lambda \in \mathfrak{t}^* \). A Verma module \( M(\Lambda) \) with highest weight \( \Lambda \) and highest weight vector \( v_\Lambda \) is a universal module with highest weight \( \Lambda \). That is, given any other highest weight module \( W \) with highest weight \( \Lambda \) and highest weight vector \( w \), there exists a unique map \( M(\Lambda) \to W \) sending \( v_\Lambda \mapsto w \).

**Remark.** Verma modules were also discovered by Harish-Chandra.

**Proposition 10.2.** Let \( \Lambda \in \mathfrak{t}^* \). Then

1. There exists a unique Verma module \( M(\Lambda) \).
2. There exists a unique irreducible highest-weight module of weight \( \Lambda \), called \( L(\Lambda) \).

**Proof.** Uniqueness of \( M(\Lambda) \) is clear from universality. For existence, take

\[
M(\Lambda) = \mathcal{U}\mathfrak{g} \otimes_{\mathfrak{u}\mathfrak{b}} \mathbb{C}_\Lambda
\]

where \( \mathfrak{b} = \mathfrak{n}^+ + \mathfrak{t} \) and \( \mathbb{C}_\Lambda \) is the \( \mathfrak{b} \)-module on which \( \mathfrak{n}^+ \cdot v = 0 \) and \( t \cdot v = \Lambda(t)v \).

**Remark.** You should think of this as the induced representation.

More concretely,

\[
M(\Lambda) = \mathcal{U}\mathfrak{g}/(\text{left ideal generated by } u - \Lambda(u) \text{ for all } u \in \mathcal{U}\mathfrak{b})
\]

(where \( \Lambda \) is extended to act on \( \mathfrak{b} \) by 0 on \( \mathfrak{n}^+ \)).

That is, \( M(\Lambda) \) is the module generated by \( \mathfrak{g} \) acting on \( 1 \) with relations \( \{ \mathfrak{n}^+ 1 = 0, t \cdot 1 = \Lambda(t)1 \text{ for all } t \in \mathfrak{t} \} = J_\Lambda \), and only the relations these imply.

If \( W \) is any other highest weight module of highest weight \( \Lambda \), then \( W = \mathcal{U}\mathfrak{g}/J \) for some \( J \supseteq J_\Lambda \), i.e. \( M(\Lambda) \twoheadrightarrow W \).

In particular, an irreducible highest-weight module must be of the form \( M(\Lambda)/I(\Lambda) \) where \( I(\Lambda) \) is a maximal proper submodule of \( M(\Lambda) \). Since we showed that there exists a unique maximal proper submodule, \( L(\Lambda) \) is unique. \( \square \)

**Proposition 10.3.** Let \( R^+ = \{ \beta_1, \ldots, \beta_r \} \). Then

\[
ed_{k_1}^{\beta_1} \ldots \ed_{k_r}^{\beta_r} v_\Lambda \text{ is a basis for } M(\Lambda).
\]

**Proof.** Immediate from the difficult direction of PBW theorem. \( \square \)

**Corollary 10.4.** Any irreducible finite-dimensional \( \mathfrak{g} \)-module is of the form \( L(\Lambda) \) for some \( \Lambda \in P^+ \).

**Proof.** We know that the highest weight must lie in \( P^+ \) by \( \mathfrak{sl}_2 \) theory. \( \square \)

**Example 10.1.** \( \mathfrak{g} = \mathfrak{sl}_2 \). The Verma module \( M(\Lambda) \) looks like this:

\[
\bullet - \bullet - \bullet - \ldots
\]

i.e. an infinite string.

**Exercise 85.**

1. Show that \( V(\lambda) = L(\lambda) \) unless \( \lambda \in \mathbb{Z}_{\geq 0} \).

2. If \( \lambda \in \mathbb{Z}_{\geq 0} \), show that \( V(\lambda) \) contains a unique proper submodule, spanned by \( F^{\lambda+1}v_\Lambda, F^{\lambda+2}v_\Lambda, \ldots \) (which is itself a Verma module). Recover the finite-dimensional irreducible representations.
Exercise 86. (1) If $\mathfrak{g}$ is an arbitrary simple Lie algebra and $\Lambda(H_i) \in \mathbb{Z}_{\geq 0}$, show that $F_i^{\Lambda(H_i)+1} \cdot v_\Lambda$ is a singular vector of $M(\Lambda)$. (There will be others!)
(2) Compute $\text{ch} M(\Lambda)$ for all Verma modules.

Lecture 19 We will now try to show that $L(\Lambda)$ is finite-dimensional if $\Lambda \in P^+$. 

Proposition 10.5. If $\Lambda \in P^+$, $L(\Lambda)$ is integrable, i.e. $E_i$ and $F_i$ act locally nilpotently.

Proof. If $V$ is any highest-weight module, $E_i$ acts locally nilpotently (as $E_iV_\Lambda \subseteq V_{\lambda+i}$ and weights of $V$ lie in $D(\Lambda)$). We must show that $F_i$ acts locally nilpotently.

By the exercise, $F_i^{\Lambda(H_i)+1} \cdot v_\Lambda$ is a singular vector. Since $L(\Lambda)$ is irreducible, it has no singular vectors other than $v_\Lambda$, so $F_i^{\Lambda(H_i)+1} v_\Lambda = 0$.

Exercise 87. (1) $a^k b = \sum_{i=0}^{k} \binom{k}{i} ((\text{ad} a)^i b) a^{k-i}$
(2) Using the above and the Serre relations $(\text{ad} e_\alpha)^4 e_\beta = 0$, show $F_i^N e_{-\beta_1} \ldots e_{-\beta_s} v_\Lambda = 0$ for $N \gg 0$ by induction on $r$.

Remark. In a general Kac-Moody algebra, the Serre relations are no longer $(\text{ad} e_\alpha)^4 e_\beta = 0$, but the exponents are still bounded above by $-a_{ij}$ for the relevant $i, j$ in the generalised Cartan matrix.

Corollary 10.6. $\dim L(\Lambda)_\mu = \dim L(\Lambda)_{w\mu}$ for all $w \in W$.

Proof. When we were proving Proposition 8.2, we only needed local nilpotence of $E_i$ and $F_i$ to conclude invariance under $w = s_{\alpha_i} = s_i$. Since $W$ is generated by $s_i$, we conclude invariance under all of $W$. \qed

Theorem 10.7 (Cartan’s theorem). If $\mathfrak{g}$ is finite-dimensional and $\Lambda \in P^+$, then $L(\Lambda)$ is finite-dimensional.

Remark. This is clear pictorially – $L(\Lambda)$ is $W$-invariant and sits in the cone $D(\Lambda)$. However, it’s faster to work through it formally.

Proof. First, we show that if $\alpha \in R^+$ then $e_{-\alpha}$ acts locally nilpotently on $L(\Lambda)$, i.e. all (not just the simple) local $\mathfrak{sl}_2$’s act integrably. In fact, $e_{-\alpha}^n v_\Lambda = 0$ for $n = (\Lambda, \alpha^\vee) + 1$, since otherwise we have $L(\Lambda)_{\Lambda-n\alpha} \neq 0$ and hence $s_\alpha(\Lambda - n\alpha)$ is a weight in $L(\Lambda)$. However, $s_\alpha(\Lambda - n\alpha) = s_\alpha(\Lambda) + n\alpha = \Lambda - (\Lambda, \alpha^\vee)\alpha + ((\Lambda, \alpha^\vee) + 1)\alpha = \Lambda + \alpha > \Lambda$ and so cannot be a weight in $L(\Lambda)$ as $\Lambda$ is the highest weight. Applying Exercise 87, we see that $e_{-\alpha}$ acts locally nilpotently on all of $L(\Lambda)$.
Therefore,

\[ L(\Lambda) = U(n^-)v_\Lambda = \{e_{-\beta_1}^{k_1} \cdots e_{-\beta_r}^{k_r}v_\Lambda\}, \quad R^+ = \{\beta_1, \ldots, \beta_r\} \]

is finite.

Lemma 10.8. \( s_i(R^+ \setminus \{\alpha_i\}) = R^+ \setminus \{\alpha_i\} \).

Proof. Let \( \alpha \in R^+ \setminus \{\alpha_i\} \), \( \alpha = \sum_{j \neq i} k_j \alpha_j + k_i \alpha_i \) with all \( k_j \geq 0 \) and some \( k_j \neq 0 \). Note that \( s_i(\alpha) \) has the same coefficients on \( \alpha_j \) for \( j \neq i \). Since \( R = R^+ \sqcup -R^+ \), we see \( s_i(\alpha) \notin -R^+ \implies s_i(\alpha) \in R^+ \).

Lemma 10.9. Recall \( \rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \). Then \( \rho(H_i) = 1 \) for all \( i \), i.e. \( \rho = \omega_1 + \ldots + \omega_l \).

Proof. Observe

\[ s_i\rho = s_i \left( \frac{1}{2} \alpha_i + \frac{1}{2} \sum_{\alpha \neq \alpha_i} \alpha \right) = -\frac{1}{2} \alpha_i + \frac{1}{2} \sum_{\alpha \neq \alpha_i} \alpha = \rho - \alpha_i. \]

On the other hand, \( s_i\rho = \rho - (\rho, \alpha_i^\vee)\alpha_i \), so \( (\rho, \alpha_i^\vee) = 1 \) for all \( i \).

Lemma 10.10 (Key lemma). Let \( \Lambda \in P^+ \), \( \mu + \rho \in P^+ \), \( \mu \leq \Lambda \). If \( |\Lambda + \rho| = |\mu + \rho| \) then \( \Lambda = \mu \).

Remark. This, again, is geometrically obvious from the fact that both \( D(\Lambda) \) and a sphere are convex.

Proof. \( 0 = (\Lambda + \rho, \Lambda + \rho) - (\mu + \rho, \mu + \rho) = (\Lambda - \mu, \Lambda + (\mu + \rho) + \rho) \)

Write \( \Lambda - \mu = \sum k_i \alpha_i \), then since \( \Lambda + (\mu + \rho) \in P^+ \) and \( (\rho, \alpha_i) > 0 \) for all \( i \), we see that we must have \( k_i = 0 \) for all \( i \).

Theorem 10.11 (Weyl complete reducibility). Let \( \mathfrak{g} \) be a semisimple Lie algebra, \( \text{char} k = 0 \), \( k = \overline{k} \). Then every finite-dimensional \( \mathfrak{g} \)-module \( V \) is a direct sum of irreducibles.

Proof. Recall that \( V \) is completely reducible as an \( (\mathfrak{sl}_2)_\alpha \)-module, \( V = \oplus V_\lambda \).

Consider \( V^{n^+} = \{x \in V : n^+x = 0\} \). By Engel’s theorem, \( V^{n^+} \neq 0 \).

Since \( [t, n^+] \subseteq n^+ \), \( t \) acts on \( V^{n^+} \), so \( V^{n^+} = \bigoplus_{\mu \in P} V^{n^+}_\mu \).

Claim. For every \( v_\mu \in V^{n^+}_\mu \), the module \( L = U\mathfrak{g} \cdot v_\mu \) is irreducible.

Proof. \( L \) is a highest weight module with highest weight \( \mu \), so we must only show that it has no other singular vectors. If \( \lambda \) is the weight of a singular vector in \( L \), then \( \lambda \leq \mu \), but \( |\lambda + \rho| = |\mu + \rho| \) by considering the action of the Casimir. Since \( V \), and therefore \( L \), is finite-dimensional, we must have \( \Lambda, \mu \in P^+ \). By the key lemma, \( \lambda = \mu \).

Consequently, \( V' = U\mathfrak{g} \cdot V^{n^+} \) is completely reducible: if \( \{v_1, \ldots, v_r\} \) is a weight basis for \( V^{n^+} \) with weights \( \lambda_1, \ldots, \lambda_r \), then \( V' = L(\lambda_1) \oplus \ldots \oplus L(\lambda_r) \). It remains to show that \( N = V/V' = 0 \).

Suppose not. \( N \) is finite-dimensional, so \( N^{n^+} \neq 0 \). Let \( v_\lambda \in N^{n^+}_\lambda \) be a singular vector with \( \lambda \in P^+ \). Lift \( v_\lambda \) to \( \overline{v_\lambda} \in V_\lambda \). Since \( v_\lambda \neq 0 \) in \( N \), we must have \( E_i \overline{v_\lambda} \neq 0 \in V_{\lambda + \alpha_i} \) for some \( i \).
Note that \( E_i v_\lambda = 0 \), so \( E_i \pi_\lambda \in V' \). Consequently, the Casimir \( \Omega \) acts on \( E_i \pi_\lambda \) by \( |\bar{\lambda} + \rho|^2 - |\rho|^2 \) for some \( \bar{\lambda} \in \{\lambda_1, \ldots, \lambda_r\} \). On the other hand, since \( v_\lambda \) was singular in \( N \), \( \Omega \) acts on \( v_\lambda \) by \( |\lambda + \rho|^2 - |\rho|^2 \). Since \( \Omega \) commutes with \( E_i \), we find \( |\bar{\lambda} + \rho| = |\lambda + \rho| \). Moreover, \( \lambda + \alpha_i \) is a weight in \( L(\bar{\lambda}) \), so \( \lambda + \alpha_i = \bar{\lambda} - \sum k_j \alpha_j \), i.e. \( \lambda < \bar{\lambda} \). This, however, contradicts the key lemma.

\[ \square \]

**Remark.** The usual proof (in particular, Weyl’s own proof) of this theorem involves lifting the integrable action to the group, and using the maximal compact subgroup (for which we have complete reducibility because it has positive, finite measure). Somehow, the key lemma is the equivalent of a positive-definite metric.

**Lemma 10.12.** Let \( \Lambda \in \mathfrak{t}^* \) and \( M(\Lambda) \) the Verma module. Then

\[ \text{ch} M(\Lambda) = \frac{e^\Lambda}{\prod_{\alpha \in R^+} (1 - e^{-\alpha})}. \]

**Lecture 20**

**Proof.** Let \( R^+ = \{\beta_1, \ldots, \beta_r\} \). By PBW, the basis for \( M(\Lambda) \) is \( \{e_{-\beta_1}^{k_1} \cdots e_{-\beta_r}^{k_r} v_\Lambda \} \) for \( k_i \in \mathbb{Z}_{\geq 0} \), and the weight of such an element is \( \Lambda - \sum k_i \beta_i \). Therefore, \( \dim M(\Lambda)_{\Lambda - \beta} \) is the number of ways of writing \( \beta \) as \( \sum k_i \beta_i \) over \( \beta_i \in R \). This is precisely the coefficient of \( e^{-\beta} \) in \( \prod_{\alpha \in R^+} (1 - e^{-\alpha})^{-1} \).

Write \( \Delta = \prod_{\alpha \in R^+} (1 - e^{-\alpha}) \). We have shown that \( \text{ch} M(\Lambda) = e^\Lambda / \Delta \).

**Lemma 10.13.** For all \( w \in W \), \( w(e^\Delta) = \det w \cdot e^\Delta \). Here, \( \det : W \to \mathbb{Z}/2 \) gives the determinant of \( w \) acting on \( \mathfrak{t}^* (\pm 1) \).

**Proof.** Since \( W \) is generated by simple reflections, it suffices to check that \( s_i(e^\Delta) = -e^\Delta \). Indeed,

\[ s_i(e^\Delta) = s_i \left( e^\Delta \prod_{\alpha \neq \alpha_i, \alpha \in R^+} (1 - e^{-\alpha}) \right) = e^{\Delta - \alpha_i} \prod_{\alpha \neq \alpha_i, \alpha \in R^+} (1 - e^{-\alpha}) = -e^\Delta. \]

\[ \square \]

**Lemma 10.14.** (1) For any highest weight module \( V(\Lambda) \) with highest weight \( \Lambda \) there exist coefficients \( a_{\lambda} \geq 0, \lambda \leq \Lambda \), such that

\[ \text{ch} V(\Lambda) = \sum_{\lambda \leq \Lambda, |\lambda + \rho| = |\Lambda + \rho|} a_{\lambda} \text{ch} L(\lambda) \]

with \( a_{\Lambda} = 1 \).

(2) There exist coefficients \( b_{\lambda} \in \mathbb{Z} \) with \( b_{\Lambda} = 1 \) such that

\[ \text{ch} L(\Lambda) = \sum_{\lambda \leq \Lambda, |\lambda + \rho| = |\Lambda + \rho|} b_{\lambda} \text{ch} M(\lambda). \]

We write \( B(\Lambda) = \{\lambda \leq \Lambda | |\lambda + \rho| = |\Lambda + \rho|\} \). Recall that \( B(\Lambda) \) is a finite set.

**Remark.** We have shown that \( B(\Lambda) \) is finite when \( \Lambda \in \mathbb{R} R \), but it is true in general.
Proof. First, note that (1) implies (2). Indeed, totally order \( B(\Lambda) \) so that \( \lambda_i \leq \lambda_j \implies i \leq j \). Applying (1) to the Verma modules gives a system of equations relating \( \text{ch} M(\lambda) \) and \( \text{ch} L(\lambda) \) which is upper-triangular with 1’s on the diagonal. Inverting this system gives (2). (Note in particular that the coefficients in (2) may be negative.)

We now prove (1). Recall that the weight spaces of a highest weight module are finite-dimensional. We induct on \( \sum_{\mu \in B(\Lambda)} \dim V(\Lambda) \mu \).

If \( V(\Lambda) \) is irreducible, we clearly have (1). Otherwise, there exists \( \mu \in B(\Lambda) \) with a singular vector \( v_\mu \in V(\Lambda) \). Pick \( \mu \) with the largest height of \( \Lambda - \mu \) (recall the height of \( \sum k_i \alpha_i \) is \( \sum k_i \)). Then \( U \mathfrak{g} \cdot v_\mu \subseteq V(\Lambda) \) has no singular vectors, and is therefore irreducible, \( L(\mu) \). Setting \( V(\Lambda) = V(\Lambda)/L(\mu) \), we see that \( V(\Lambda) \) is a highest-weight module with a smaller value of \( \sum_{\mu \in B(\Lambda)} \dim V(\Lambda) \mu \), and \( \text{ch} V(\Lambda) = \text{ch} V(\Lambda) + \text{ch} L(\mu) \). \( \square \)

We will now compute \( \text{ch} L(\Lambda) \) for \( \Lambda \in P^+ \).

We know \( \text{ch} L(\Lambda) = \sum_{\lambda \in B(\Lambda)} b_\lambda \text{ch} M(\lambda) = \sum_{\lambda \in B(\Lambda)} b_\lambda e^\lambda/\Delta \).

Now, \( w(\text{ch} L(\Lambda)) = \text{ch} L(\Lambda) \) for all \( w \in W \), and \( w(\Delta e^\rho) = \det w \cdot \Delta e^\rho \). Therefore,

\[
e^\rho \Delta \text{ch} L(\Lambda) = \sum_{\lambda \in B(\Lambda)} b_\lambda e^{\lambda + \rho}
\]

is \( W \)-antiinvariant. We can therefore rewrite

\[
\sum_{\lambda \in B(\Lambda)} b_\lambda e^{\lambda + \rho} = \sum_{\text{orbits of } W \text{ on } B(\Lambda) + \rho} \sum_{w \in W} b_\lambda \det w \cdot e^{w(\lambda + \rho)}.
\]

Now, if \( \lambda \in \mathbb{R}R \) (which is true, since \( \Lambda \in P^+ \)), then \( W(\lambda + \rho) \) intersects \( \{ x \in \mathbb{R}R | (x, \alpha_i^\vee) \geq 0 \} \) in exactly one point (this set is a fundamental domain for the \( W \)-action, and we won’t have boundary problems because \( (\rho, \alpha_i^\vee) = 1 > 0 \)). Therefore, we can take a dominant weight as the orbit representative. Note, however, that there is only one dominant weight in \( B(\Lambda) \), namely, \( \Lambda! \) (This was the Key Lemma 10.10.) Therefore, we derive

**Theorem 10.15 (Weyl Character Formula).**

\[
\text{ch} L(\Lambda) = \sum_{w \in W} \det w \cdot e^{w(\Lambda + \rho)} = \sum_{w \in W} \det w \cdot \text{ch} M(w(\Lambda + \rho) - \rho).
\]

**Example 10.2.** Let \( \mathfrak{g} = \mathfrak{sl}_2 \), and write \( z = e^{\alpha/2} \). Then \( \mathbb{C}[P] = \mathbb{C}[z, z^{-1}] \) and \( \rho = z \). We have

\[
\text{ch} L(m \alpha/2) = (z^{m+1} - z^{-(m+1)})/(z - z^{-1})
\]
as before.

**Remark.** The second expression for the character of \( L(\Lambda) \) looks like an Euler characteristic. In fact, it is: there is a complex of Verma modules whose cohomology are the irreducible modules. This is known as the “Bernstein-Gelfand-Gelfand” resolution.

The first expression comes from something like a cohomology of coherent line bundles, and is variously known as Euler-Grothendieck, Atiyah-Bott, and Atiyah-Singer.
Corollary 10.16. Since $L(0) = \mathbb{C}$, we must have $\text{ch} L(0) = 1$, so

$$e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha}) = \sum_{w \in W} \det w \cdot e^{\lambda w}.$$ 

This is known as the Weyl denominator identity.

Exercise 88. Let $g = \mathfrak{sl}_n$. Show that the Weyl denominator identity is equivalent to the Vandermonde determinant

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1} \end{pmatrix} = \prod_{i<j} (z_j - z_i),$$

where we write $z_i = e^{-\alpha_i}$.

Corollary 10.17 (Weyl dimension formula).

$$\dim L(\Lambda) = \prod_{\alpha \in R^+} (\alpha, \Lambda + \rho)/(\alpha, \rho).$$

Example 10.3. $g = \mathfrak{sl}_3$ of type $A_2$, $R^+ = \{\alpha, \beta, \alpha + \beta\}$ with $\rho = \alpha + \beta = \omega_1 + \omega_2$. Let $\Lambda = m_1 \omega_1 + m_2 \omega_2$, then

$$\begin{array}{c|c|c|c}
\langle \cdot, \Lambda + \rho \rangle & \alpha & \beta & \alpha + \beta \\
\langle \cdot, \rho \rangle & m_1 + 1 & m_2 + 1 & m_1 + m_2 + 2 \\
\end{array}$$

Therefore, the dimension of $L(\Lambda)$ is $1/2(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)$.

“Just to check that you too certainly know the dimensions of the irreducible representations, let me state the obvious exercise.”

Exercise 89. Compute the dimensions of all the finite-dimensional irreducible representations of $B_2$ and $G_2$.

Lecture 21

Remark. Let $w \in W$ be written as $w = s_{i_1}s_{i_2}\ldots s_{i_r}$, where $s_{i_k}$ are simple reflections. Then $\det w = (-1)^r$. The minimal $r$ such that $w$ can be written in this form is called the length $l(w)$ of $w$, denoted $l(w)$. The Monoid Lemma asserts that you can get from one minimal-length expression for $w$ to another by repeatedly applying the braid relations.

Exercise 90. $l(w) = \#\{-R^+ \cap w^{-1}R^+\} = l(w^{-1})$.

We now go back to our proof.

Proof. We know $\text{ch} L(\Lambda) = \sum \dim L(\Lambda) \lambda e^\lambda \in \mathbb{C}[P]$. We would like to set $e^\lambda \mapsto 1$, but that leads to a $0/0$ expression in the Weyl character formula, which is hard to evaluate. Therefore, we will start by defining the homomorphism $F_\mu : \mathbb{C}[P] \to \mathbb{C}(q)$, $e^\lambda \mapsto q^{-\langle \lambda, \mu \rangle}$. We would like to evaluate $F_\mu(\text{ch} L(\lambda))$, since $F_\mu(e^\mu) = 1$ for all $\mu$.

We apply $F_\mu$ to the Weyl denominator identity:

$$q^{-\langle \rho, \mu \rangle} \prod_{\alpha \in R^+} (1 - q^{\langle \alpha, \mu \rangle}) = \sum_{w \in W} \det w q^{-\langle \rho, w \cdot \mu \rangle} = \sum_{w \in W} \det w q^{-\langle \rho, w^{-1} \cdot \mu \rangle}$$
since \( \det w = \det w^{-1} \) and \((x, wy) = (w^{-1}x, y)\) (i.e. the Weyl group is a subgroup of the orthogonal group of the inner product).

We now apply \( F_\mu \) to the Weyl character formula:

\[
F_\mu(\text{ch } L(\lambda)) = \sum_{w \in W} \det w q^{-(w(\lambda+\rho),\mu)}q^{-(\rho,\mu)} \prod_{\alpha \in R^+} (1-q^{(\alpha,\mu)})
\]

provided \((\alpha, \mu) \neq 0\) for all \(\alpha \in R^+\).

Take \(\mu = \rho\) (recall that \((\rho, \alpha_i) > 0\) for all simple roots \(\alpha_i\), so \((\rho, \alpha) > 0\) for all \(\alpha \in R^+\)). Then

\[
F_\rho(\text{ch } L(\lambda)) = \sum \dim L(\lambda) q^{-(\lambda,\rho)} = \frac{q^{-(\rho,\Lambda+\rho)} \prod_{\alpha \in R^+} (1-q^{(\alpha,\Lambda+\rho)})}{q^{-(\rho,\rho)} \prod_{\alpha \in R^+} (1-q^{(\alpha,\alpha)})}
\]

where we used our expression for the Weyl denominator identity and applied it to the numerator.

Setting \(q = 1\) and applying L'Hôpital's rule,

\[
\dim L(\Lambda) = \prod_{\alpha \in R^+} \frac{(\Lambda + \rho, \alpha)}{(\rho, \alpha)}.
\]

\[ \square \]

**Remark.** “You are now in a position to answer any question you choose to answer, if necessary by brute force.” For example, let us decompose \( L(\lambda) \otimes L(\mu) = \sum m^\nu_{\lambda\mu} L(\nu) \); we want to compute the Littlewood-Richardson coefficients \( m^\nu_{\lambda\mu} \) (recall that we had the Clebsch-Gordan rule for them in \( sl_2 \)).

Define \(-: \mathbb{Z}[P] \to \mathbb{Z}[P], e^\lambda \mapsto e^{-\lambda}, \) and \( CT : \mathbb{Z}[P] \to \mathbb{Z} \) with \(e^\lambda \mapsto 0 \) unless \(\lambda = 0\), and \(e^0 \mapsto 1\). Define \((,): \mathbb{Z}[P] \times \mathbb{Z}[P] \to \mathbb{Z} \) by \((f,g) = \frac{1}{|W|} CT(fg\Delta\Sigma)\), where \(\Delta = \prod_{\alpha \in R^+} (1-e^{-\alpha})\).

**Claim.** Let \(\chi_\lambda = \text{ch } L(\lambda), \chi_\mu = \text{ch } L(\mu)\), then \((\chi_\lambda, \chi_\mu) = \delta_{\mu\lambda}. \) (If our Lie algebra has type \(A\), then \(\chi_\lambda\) are the Schur functions, and this is their orthogonality.)

Given this claim, we have \(m^\nu_{\lambda\mu} = (\chi_\lambda \chi_\mu, \chi_\nu)\) is algorithmically computable. It was a great surprise to everyone when it was discovered that actually there is a simpler and more beautiful way of doing this.

**Proof of claim.**

\[
(\chi_\lambda, \chi_\mu) = \frac{1}{|W|} CT(\sum_{x,w \in W} e^{w(\lambda+\rho)-\rho} e^{x(\mu+\rho)-\rho} \det(wx))
\]

However, \(CT(e^{w(\lambda+\rho)-x(\mu+\rho)}) = \delta_{wx}\delta_{\mu\lambda}\), since for \(\lambda, \mu \in R^+\) we have \(x^{-1}w(\lambda+\rho) = \mu+\rho \Rightarrow x = w, \lambda = \mu\). \[ \square \]

11. **Principal \( sl_2 \)**

Define \(\rho^L \in t^*\) by \((\rho^L, \alpha_i) = 1\) for all \(\alpha_i \in \Pi\). (Compare with \(\rho\), which had \((\rho, \alpha_i^\vee) = 1\).)

**Exercise 91.** Show \(\rho^L = \frac{1}{2} \sum_{\alpha \in R^+} \alpha^\vee\). In particular, if \(R\) is simply laced, \(\rho = \rho^L\).

Then \((\rho^L, \alpha) = \text{ht}(\alpha) = \sum k_i\) if we write \(\alpha = \sum k_i \alpha_i\).
Exercise 92.

\[ F_{\rho L}(\text{ch} \ L(\Lambda)) = q^{-(\Lambda, \rho L)} \prod_{\alpha \in R^+} \frac{(1 - q^{(\Lambda + \rho, \alpha^\vee)})}{(1 - q^{\rho, \alpha^\vee})}. \]

[Hint: apply \( F_{\rho L} \) to the Weyl denominator identity for the Langlands dual Lie algebra with root system \( R' \).]

Note that \((\lambda + \rho, \alpha^\vee)/(\rho, \alpha^\vee) = (\lambda + \rho, \alpha)/(\rho, \alpha)\) as \( \alpha^\vee = 2\alpha/(\alpha, \alpha) \), so we recoved the same formula for Weyl dimension.

**Definition.** We call \( F_{\rho L}(\text{ch} \ L(\Lambda)) \) the \( q \)-dimension of \( L(\lambda) \), denoted \( \dim_q L(\Lambda) \).

**Proposition 11.1.** \( \dim_q L(\Lambda) \) is a unimodal polynomial. More specifically, it lives in \( \mathbb{N}[q^2, q^{-2}]^{\mathbb{Z}/2} \) or \( q\mathbb{N}[q^2, q^{-2}]^{\mathbb{Z}/2} \) (depending on its degree), and the coefficients decrease as the absolute value of the degree gets bigger.

**Proof.** We will show that \( \dim_q L(\Lambda) \) is the character of an \( \mathfrak{sl}_2 \)-module in which all “strings” have the same parity.

Set \( H = 2\nu^{-1}(\rho L) \in \mathfrak{t} \subseteq \mathfrak{g} \), and set \( E = \sum E_i \).

**Exercise 93.** Check that \([H, E] = 2E\). Writing \( H = \sum c_i H_i \) with \( c_i \in \mathbb{C} \), set \( F = \sum c_i F_i \).

**Exercise 94.**

1. Show that \( E, F, H \) generate a copy of \( \mathfrak{sl}_2 \). This is called the principal \( \mathfrak{sl}_2 \).
2. Show that if \( \Lambda - \gamma \) is a weight of \( L(\Lambda) \), then \((\Lambda - \gamma, 2\rho L) \equiv (\Lambda, 2\rho L) \mod 2\).
3. Derive the proposition.

**Exercise 95.** Write \([n] = (q^n - 1)/(q - 1)\), and \([n]! = [n][n - 1]\ldots[1]\). Show that the following polynomials are unimodal:

\[ \binom{n}{k} = \frac{[n]!}{[k]![n-k]!} \frac{(1 + q)(1 + q^2)\ldots(1 + q^n)}{ (1 + q^k)(1 + q^{k+1})\ldots(1 + q^{n-k})} \]

[Hint: for the first, apply the above arguments to \( \mathfrak{g} = \mathfrak{sl}_n \) and \( V = S^k \mathbb{C}^n \) or \( \Lambda^k \mathbb{C}^{n+k} \). For the second, apply it to the spin representation of \( B_n = \mathfrak{so}_{2n+1} \); we will define the spin representation next time.]

**Remark.** When is \( L(\lambda) \cong L(\lambda)^* \)? Precisely when the lowest weight of \( L(\lambda) \) is \(-\lambda\). The lowest weight of \( L(\lambda) \) is \( w_0 \lambda \), where \( w_0 \) is the (unique) maximal-length element of \( W \) which sends all the positive roots to negative ones.

Now, for a representation \( V \) which is isomorphic to its dual \( V^* \), we get a bilinear form on \( V \). This bilinear form is either symmetric or antisymmetric. By above, we see that we can determine which it is by laboriously computing characters; but can we tell simply by looking at \( \lambda \)?

The answer turns out to be yes. Indeed, the form will pair the highest-weight and the lowest-weight vector, and if we look at the principal \( \mathfrak{sl}_2 \), it is a string from the highest-weight to the lowest-weight vector. Now, on the principal \( \mathfrak{sl}_2 \) it is quite easy to see that the form is antisymmetric if the representation is \( L(n) \) with \( n \) odd, and symmetric if it is \( L(n) \) with \( n \) even. Thus, all we need to know is (the parity of) \((\lambda, 2\rho L)\). This is called the Schur indicator.
Lecture 22

Exercise 96. Compute dim_q L(θ), where L(θ) is the adjoint representation, for G_2, A_2, and B_2. Then do it for all the classical groups. You will notice that L(θ)|principal sl_2 = L(2e_1) + \ldots + L(2e_l) where l = rank g = dim t, e_1, \ldots, e_l \in \mathbb{N}, and e_1 = 1. The e_i are called the exponents of the Weyl group, and the order of the Weyl group is |W| = (e_1+1) \ldots (e_l+1).

Compute |W| for E_8.

12. Crystals

Let g be a semisimple Lie algebra, Π = {α_1, \ldots, α_l} the simple roots, and P the weight lattice.

Definition. A crystal is a set B, 0 \not\in B, together with functions wt : B \to P, \tilde{e}_i : B \to B \sqcup \{0\}, \tilde{f}_i : B \to B \sqcup \{0\} such that

1. If \tilde{e}_i b \neq 0, then wt(\tilde{e}_i b) = wt(b) + \alpha_i; if \tilde{f}_i b \neq 0, then wt(\tilde{f}_i b) = wt(b) - \alpha_i.
2. For b, b' \in B, \tilde{e}_i b = b' if and only if b = \tilde{f}_i b'.
3. Before we state the third defining property, we need to introduce slightly more notation.

We can draw B as a graph. The vertices are b \in B, and the edges are b \to b' if \tilde{e}_i b = b'. We say that this edge is “coloured by i”. This graph is known as the crystal graph.

Example 12.1. In sl_2 the string

\[ n \to n - 2 \to n - 4 \to \ldots \to -n \]

is a crystal, where the weight of vertex i is iα/2.

Define \epsilon_i(b) = \max \{n \geq 0 : \tilde{e}_i^n(b) \neq 0\}, and \phi_i(b) = \max \{n \geq 0 : \tilde{f}_i^n(b) \neq 0\}.

\[
\bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet, \quad \epsilon_i \quad \phi_i
\]

In sl_2, the sum \epsilon_i(b) + \phi_i(b) is the length of the string. On the other hand, if we are in the highest-weight representation L(n) = L(nw) and wt(b) = n - 2k, then \epsilon(b) = k and \phi(b) = n - k, i.e. \phi(b) - \epsilon(b) = (wt(b), \alpha_i^\vee). The third property of the Lie algebras is that this happens in general:

3. \phi_i(b) - \epsilon_i(b) = (wt(b), \alpha_i^\vee) for all \alpha_i \in Π.

Define B_μ = \{b \in B : wt(b) = μ\}.

Definition. For B_1 and B_2 crystals, we define the tensor product B_1 \otimes B_2 as follows: as a set, B_1 \otimes B_2 = B_1 \times B_2, wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2), and

\[
\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} 
(\tilde{e}_i b_1) \otimes b_2, & \text{if } \phi_i(b_1) \geq \epsilon_i(b_2) \\
0, & \text{if } \phi_i(b_1) < \epsilon_i(b_2)
\end{cases} \quad \tilde{f}_i(b_1 \otimes b_2) = \begin{cases} 
(\tilde{f}_i b_1) \otimes b_2, & \text{if } \phi_i(b_1) > \epsilon_i(b_2) \\
(\tilde{f}_i b_1) \otimes (\tilde{f}_i b_2), & \text{if } \phi_i(b_1) \leq \epsilon_i(b_2)
\end{cases}
\]

That is, we are trying to recreate the picture in Figure 10 in each colour.

Exercise 97. Check that B_1 \otimes B_2 as defined above is a crystal.

Exercise 98. B_1 \otimes (B_2 \otimes B_3) \cong (B_1 \otimes B_2) \otimes B_3 via b_1 \otimes (b_2 \otimes b_3) \mapsto (b_1 \otimes b_2) \otimes b_3.
Remark. As we see above, the tensor product is associative. In general, however, it is not commutative.

Definition. $B^\vee$ is the crystal obtained from $B$ by reversing the arrows. That is, $B^\vee = \{ b^\vee : b \in B \}$, $wt(b^\vee) = -wt(b)$, $e_i(b^\vee) = \phi_i(b)$ (and vice versa), and $\tilde{e}_i(b^\vee) = (\tilde{f}_i b)^\vee$ (and vice versa).

Exercise 99. $(B_1 \otimes B_2)^\vee = B_2^\vee \otimes B_1^\vee$.

Remark. We will see in a moment precisely how we can attach a crystal to the basis of a representation. If $B$ parametrises a basis for $V$, then $B^\vee$ parametrises a basis for $V^\ast$. The condition on the weights can be seen by noting that if $L(\lambda)$ has highest weight $\lambda$, then $L(\lambda)^\ast$ has lowest weight $-\lambda$.

Theorem 12.1 (Kashiwara). Let $L(\lambda)$ be an irreducible highest-weight representation with highest weight $\lambda \in P^\ast$. Then

1. There exists a crystal $B(\lambda)$ whose elements are in one-to-one correspondence with a basis of $L(\lambda)$, and elements of $B(\lambda)_\mu$ parametrise $L(\lambda)_\mu$, so that $\text{ch} L(\lambda) = \sum_{b \in B(\lambda)} e^{wt(b)}$.

2. For any simple root $\alpha_i$ (i.e. a simple $\mathfrak{sl}_2 \subseteq \mathfrak{g}$), the decomposition of $L(\lambda)$ as an $(\mathfrak{sl}_2)_i$-module is given by the $i$-coloured strings in $B(\lambda)$. (In particular, as an uncoloured graph, $B(\lambda)$ is connected, since it is spanned by $(\prod \tilde{f}_i) v_\lambda$.)

3. $B(\lambda) \otimes B(\mu)$ is the crystal for $L(\lambda) \otimes L(\mu)$ i.e. $B(\lambda) \otimes B(\mu)$ decomposes into connected components in the same way as $L(\lambda) \otimes L(\mu)$ decomposes into irreducible representations.

Remark. We will mention three proof approaches below. The first of them is due to Kashiwara, and is in the lecturer’s opinion the most instructive.

Example 12.2. Let $\mathfrak{g} = \mathfrak{sl}_3$, $V = \mathbb{C}^3 = L(\omega_1)$. We compute $V \otimes V$ and $V \otimes V^\ast$.

Remark. Note that while the crystals give the decomposition of the representation into irreducibles, they do not correspond directly to a basis. That is, there is no $\mathfrak{sl}_2$-invariant basis that we could use here. Kashiwara’s proof of the theorem uses the quantum group $U_q \mathfrak{g}$, which is an algebra over $\mathbb{C}[q, q^{-1}]$ and a deformation of the universal enveloping algebra $U \mathfrak{g}$. The two have the same representations, but over $\mathbb{C}[q, q^{-1}]$ there is a very nice basis which satisfies $e_i b = \tilde{e}_i b + q \cdot (\text{some mess})$. Therefore, setting $q = 0$ (“freezing”) will give the crystal.
Lusztig’s proof uses some of the same ideas, where I didn’t catch how it goes.

Soon, we will look at Littelmann paths, which give a purely combinatorial way of proving this theorem (which, on the face of it, is a purely combinatorial statement).

**Definition.** A crystal is called **integrable** if it is a crystal of a highest-weight module with highest weight $\lambda \in \mathbb{P}^+$.

For two integrable crystals $B_1, B_2$, we do in fact have $B_1 \otimes B_2 = B_2 \otimes B_1$ (in general, this is false).

**Remark.** There is a combinatorial condition due to Stembridge which determines whether a crystal is integrable; it is a degeneration of the Serre relations, but we will not state it precisely.

**Lecture 23** Consider the crystal for the standard representation of $\mathfrak{sl}_n$, $L(\omega_1) = \mathbb{C}^n$:

$$
\begin{align*}
\omega_1 & \rightarrow \omega_1 - \alpha_1 \\
2\omega_1 & \rightarrow \omega_1 - \alpha_1 \\
2\omega_1 - \alpha_1 & \rightarrow \omega_1 - \alpha_1 - \alpha_2 \\
& \quad \vdots \\
\omega_1 - \alpha_1 - \cdots - \alpha_n & \rightarrow \n-1 \\
\end{align*}
$$

We can use this to construct the crystals for all representations of $\mathfrak{sl}_n$, as follows:

Let $\lambda \in \mathbb{P}^+$, $\lambda = k_1 \omega_1 + \ldots + k_n \omega_n$. Then $L(\lambda)$ must be a summand of $L(\omega_1)^{\otimes k_1} \otimes \ldots \otimes L(\omega_n)^{\otimes k_n}$, since the highest weight of this representation is $\lambda$ (with highest weight vector $\otimes v_{\omega_i}^{\otimes k_i}$, where $v_{\omega_i}$ was the highest-weight vector of $L(\omega_i)$). Moreover, $L(\omega_i) = \Lambda^i \mathbb{C}^n$ is a summand of $(\mathbb{C}^n)^{\otimes i}$. We conclude that $L(\lambda)$ occurs in some $(\mathbb{C}^n)^{\otimes N}$ for $N > 0$. Therefore, the crystal for $\mathbb{C}^n$ together with the rule for taking tensor products of crystals determines the crystal of every representation of $\mathfrak{sl}_n$.

We introduce the **semi-standard Young tableau** of a representation. (This is due to Hodge, Schur, and Young.) Write

$$
B(\omega_1) = \begin{array}{cccc}
1 & 2 & 3 & \cdots & n \\
\end{array}
$$

for the crystal of the standard representation of $\mathbb{C}^n$. For $i < n$ let

$$
b_i = \begin{array}{cccc}
1 & 2 & \cdots & i \\
\end{array} \in B(\omega_1)^{\otimes i}.
$$

(This corresponds to the vector $v_1 \wedge v_2 \wedge \ldots \wedge v_i \in \Lambda^i \mathbb{C}^n$, where $v_k$ are the basis vectors of $\mathbb{C}^n$.)

**Exercise 100.** (1) $b_i$ is a highest-weight vector in $B(\omega_1)^{\otimes i}$ of weight $\omega_i = e_1 + \ldots + e_i$.

(Recall that $b \in B$ is a highest-weight vector if $e_i b = 0$ for all $i$.) Hence, the connected component of $B(\omega_1)^{\otimes i}$ containing $b_i$ is $B(\omega_i)$. 

![Figure 11. Crystals for $V \otimes V$ and $V \otimes V^*$. Observe $V \otimes V = S^2 V + \Lambda^2 V = S^2 V + V^*$, and $V \otimes V^* = \mathbb{C} + \mathfrak{sl}_3$.](image)
(2) The connected component of $B(\omega_i)$ consists precisely of
\[
\{a_1 \otimes a_2 \otimes \ldots \otimes a_i | 1 \leq a_1 < \ldots < a_i \leq n \} \subset B(\omega_1)^{\otimes i}.
\]

We write elements of the form $a_1 \otimes a_2 \otimes \ldots \otimes a_i$ as column vectors $\begin{bmatrix} a_1 & a_2 & \cdots & a_i \end{bmatrix}$. For example, the highest weight vector is denoted $\begin{bmatrix} 1 & 2 & \cdots & n \end{bmatrix}$.

Let $\lambda = \sum k_i \omega_i$. Embed $B(\lambda) \hookrightarrow B(\omega_1)^{\otimes k_1} \otimes B(\omega_2)^{\otimes k_2} \otimes \ldots \otimes B(\omega_{n-1})^{\otimes k_{n-1}}$ by mapping the highest weight vector $b_\lambda \mapsto b_1^{\otimes k_1} \otimes \ldots \otimes b_{n-1}^{\otimes k_{n-1}}$ (which is a highest-weight vector in the image). Now, we can represent any element in $B(\omega_1)^{\otimes k_1} \otimes \ldots \otimes B(\omega_{n-1})^{\otimes k_{n-1}}$ by a sequence of column vectors as in Figure 12, where the entries in the boxes are strictly increasing down columns and (non-strictly) decreasing along rows.

**Figure 12.** Generic element of $B(\omega_1)^{\otimes k_1} \otimes \ldots \otimes B(\omega_{n-1})^{\otimes k_{n-1}}$, aka a semi-standard Young tableau. Here, $k_{n-1} = 3, k_{n-2} = 2, k_{n-3} = 1, \ldots, k_2 = 4, k_1 = 3$.

**Definition.** A *semi-standard Young tableau of shape* $\lambda$ is an array of numbers with dimensions as above, such that the numbers strictly increase down columns, and (non-strictly) decrease along rows.

**Theorem 12.2** (Exercise). (1) The connected component of $B(\lambda)$ in $B(\omega_1)^{\otimes k_1} \otimes B(\omega_{n-1})^{\otimes k_{n-1}}$ are precisely the semi-standard Young tableaux of shape $\lambda$.

(2) Describe the action of $\tilde{e}_i$ and $\tilde{f}_i$ explicitly in terms of tableaux.

We will now construct the Young tableaux for all the classical Lie algebras.

**Example 12.3.**

- $\mathfrak{so}_{2n+1}$, i.e. type $B_n$: for the standard representation $\mathbb{C}^{2n+1}$ we have the crystal

\[
\begin{array}{cccccccc}
1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & \ldots & \rightarrow & n \rightarrow 0 \\
1 & \rightarrow & n & \rightarrow & 0 & \rightarrow & \ldots & \rightarrow & 2 \rightarrow 1
\end{array}
\]
• $\mathfrak{so}_{2n}$ (type $D_n$): the crystal of the standard representation $\mathbb{C}^{2n}$ is

\[
\begin{array}{cccccccc}
1 & 1 & \rightarrow & 2 & 2 & \rightarrow & \ldots & n-2 & n-1 \\
& n & \rightarrow & & & \rightarrow & & & n \rightarrow n-1 \\
& n & \rightarrow & & & \rightarrow & & & n \rightarrow n-1 \\
\end{array}
\]

Exercise 101.  
(1) Check that the crystals of the standard representations are as claimed. 
(2) What subcategory of the category of representations of $\mathfrak{g}$ do these representations generate? 
Consider the highest weight $\lambda$ of the standard representation. This gives an element $\bar{\lambda} \in P/Q = Z(G)$, a finite group (where $G$ is the simply connected Lie group attached to $\mathfrak{g}$). Consider the subgroup $\langle \bar{\lambda} \rangle \leq P/Q$. We cannot obtain all of the representations unless $P/Q$ is cyclic and generated by $\bar{\lambda}$. For the classical examples we have $P/Q = \mathbb{Z}/2 \times \mathbb{Z}/2$ for $D_{2n}$, $\mathbb{Z}/4$ for $D_{2n+1}$, and $\mathbb{Z}/2$ for $B_n$ and $D_{2n}$.
(3) (Optional) Write down a combinatorial set like Young tableaus that is the crystal of $B(\lambda)$ with $\lambda$ obtained from the standard representation.

For $B_n$ we have one more representation, the spin representation. Recall the Dynkin diagram for $B_n$,

\[
B_n \quad e_{n-1} - e_n \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\]

Definition. Let $\omega_1, \ldots, \omega_n$ be the dual basis to $\alpha_1, \ldots, \alpha_n$. We call $L(\omega_n)$ the spin representation.

Exercise 102. Use the Weyl dimension formula to show that $\dim L(\omega_n) = 2^n$.

Define $B = \{(i_1, \ldots, i_n)|i_j = \pm 1\}$ with wt$((i_1, \ldots, i_n) = \frac{1}{2} \sum i_j e_j \in P$, and

\[
e_i(i_1, \ldots, i_n) = \begin{cases} 
(i_1, \ldots, +1_j, -1_{j+1}, \ldots, i_n), & \text{if } (i_j, i_{j+1}) = (-1, +1), 1 \leq j \leq n-1 \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
e_n(i_1, \ldots, i_n) = \begin{cases} 
(i_1, \ldots, i_{n-1}, 1), & \text{if } i_n = -1 \\
0, & \text{otherwise}
\end{cases}
\]

(note that in particular $e_i^n = 0$ always).

Claim. This is the crystal of the spin representation $L(\omega_n)$.

Remark. Note that $\dim L(\omega_n) = \dim \Lambda \mathbb{C}^n$. This is for a very good reason. Indeed, $\mathfrak{gl}_n \subset \mathfrak{so}_{2n+1}$ via $A \mapsto \begin{pmatrix} A & 0 \\ 0 & -JA^T J^{-1} \end{pmatrix}$, and $L(\omega_n)_{|\mathfrak{gl}_n} = \Lambda \mathbb{C}^n$. 
Exercise 103. Check that $B|_{\mathfrak{sl}_n}$ is the crystal of $\Lambda C^n$.

For type $D_n$, the situation is more complicated. We can define representations

$$V^+ = L(\omega_n), \quad V^- = L(\omega_{n-1});$$

these are called the half-spin representations. The corresponding crystals are $B^\pm = \{(i_1, \ldots, i_n)|i_j = \pm 1\}$ with $\prod i_j = 1$ for $B^+$ and $\prod i_j = -1$ for $B^-$. We define the weight and most $\tilde{e}_i, \tilde{f}_i$ as before, with the change that

$$\tilde{e}_n(i_1, \ldots, i_n) = \begin{cases} (i_1, \ldots, i_{n-2}, +1, +1) & \text{if } (i_{n-1}, i_n) = (-1, -1) \\ 0, & \text{otherwise.} \end{cases}$$

Lecture 24

13. Littelmann Paths

Set $P_\mathbb{R} = P \otimes \mathbb{Z} \mathbb{R}$. By a path we mean a piecewise linear continuous map $[0, 1] \to P_\mathbb{R}$. We consider paths up to reparametrisation, i.e. $\pi \cong \pi \circ \phi$, where $\phi : [0, 1] \to [0, 1]$ is a piecewise-linear isomorphism. (That is, we care about the trajectory, not about the speed at which we traverse the path.)

Let $\mathcal{P} = \{\text{paths } \pi \text{ such that } \pi(0) = 0, \pi(1) \in P\}$. Define a crystal structure on $\mathcal{P}$ as follows: we set $wt(\pi) = \pi(1)$.

To define $\tilde{e}_i(\pi)$, let

$$h_i = \min \mathbb{Z} \cap \{\langle \pi(t), \alpha_i^\vee \rangle|0 \leq t \leq 1\} \leq 0.$$ 

That is, $h_i$ is the smallest integer in $\langle \pi(t), \alpha_i^\vee \rangle$ (note that since $\pi(0) = 0$, we have $h_i \leq 0$).

If $h_i = 0$, we set $\tilde{e}_i(\pi) = 0$ (this is not the path that stays at 0, but rather the extra element in the crystal).

If $h_i < 0$, take the smallest $t_1 > 0$ such that $\langle \pi(t_1), \alpha_i^\vee \rangle = h_i$. Then take the largest $t_0 < t_1$ such that $\langle \pi(t_0), \alpha_i^\vee \rangle = h_i + 1$.

The idea is to reflect the segment of the path $\pi$ from $t_0$ to $t_1$ in the $\langle \lambda, \alpha_i^\vee \rangle = h_i + 1$ plane. Therefore, we define the path $\tilde{e}_i(\pi)$ as follows:

$$\tilde{e}_i(\pi)(t) = \begin{cases} \pi(t), & 0 \leq t \leq t_0 \\ \pi(t_0) + s_{\alpha_i}(\pi(t) - \pi(t_0)) = \pi(t) - \langle \pi(t) - \pi(t_0), \alpha_i^\vee \rangle \alpha_i, & t_0 \leq t \leq t_1 \\ \pi(t) + \alpha_i, & t \geq t_1 \end{cases}$$

See Figure 13.

Exercise 104. Show that $e_i(\pi) = -h_i$.

Example 13.1. Let’s compute the crystals of some representations of $\mathfrak{sl}_2$:

$$\tilde{e}_i(\bullet_{-\alpha_i/2} \leftarrow \bullet) = (\bullet \longrightarrow_{\alpha_i} \bullet_{-\alpha_i/2}), \quad \tilde{e}_i(\bullet_{-\alpha_i} \leftarrow \bullet) = 0$$

and

$$\tilde{e}_i(-\alpha_i \leftarrow \bullet \leftarrow \bullet) = (-\bullet \longrightarrow_{\alpha_i} \bullet_{-\alpha_i/2}) \quad \tilde{e}_i(-\bullet \leftarrow \bullet_{-\alpha_i/2} \longrightarrow \bullet) = (\bullet \longrightarrow \bullet \longrightarrow \bullet_{-\alpha_i})$$

If $\pi$ is a path, let $\pi^\vee$ be the reversed path, i.e. $t \mapsto \pi(1 - t) - \pi(1)$. Define

$$\tilde{f}_i(\pi) = (\tilde{e}_i(\pi^\vee))^\vee.$$ 

Exercise 105. With the above definitions, $\mathcal{P}$ is a crystal.
Define $\mathcal{P}^+ = \{\text{paths } \pi \text{ such that } \pi[0, 1] \subset P_\mathbb{R}^+ = \{x \in P_\mathbb{R} : \langle x, \alpha_i^\vee \rangle \geq 0 \text{ for all } i\}\}$. If $\pi \in \mathcal{P}^+$, then $\tilde{e}_i(\pi) = 0$ for all $i$.

For $\pi \in \mathcal{P}^+$ let $B_\pi$ be the subcrystal of $\mathcal{P}$ generated by $\pi$ (i.e. $B_\pi = \{\tilde{f}_i, \tilde{f}_{i_2} \ldots \tilde{f}_i, \pi\}$).

**Theorem 13.1** (Littelmann).  
(1) If $\pi, \pi' \in \mathcal{P}^+$ then $B_\pi \cong B_{\pi'}$ as crystals iff $\pi(1) = \pi'(1)$.

(2) There is a unique isomorphism between this crystal and $B(\pi(1))$, the crystal of the irreducible representation $L(\pi(1))$. (Of course, the isomorphism sends $\pi$ to $\pi(1)$.)

(3) Moreover, for the path $\pi(t) = \lambda t$, $\lambda \in \mathcal{P}^+$, Littelmann gives an explicit combinatorial description of the paths in $B_\pi$.

**Example 13.2.** Let’s compute the crystal of the adjoint representation of $\mathfrak{sl}_3$:

$$
\tilde{e}_\beta \begin{pmatrix} \bullet \\ - (\alpha + \beta) \end{pmatrix} = \begin{pmatrix} \bullet \\ - \alpha \end{pmatrix}
$$

**Exercise 106.** Compute the rest of the crystal, and check that you get the adjoint representation of $\mathfrak{sl}_3$.

The simplest nontrivial example is $G_2$, the automorphisms of octonions.

**Exercise 107.** Compute the crystal of the 7-dimensional and the 14-dimensional (adjoint) representation of $G_2$. Compute their tensor product.

The tensor product of crystals has a very nice (and natural!) realisation in terms of concatenating paths:

**Definition.** For $\pi_1, \pi_2 \in \mathcal{P}$, define $\pi_1 \ast \pi_2$ to be the concatenation (traverse $\pi_1$, then translate $\pi_2$ to start at $\pi_1(1)$ and traverse it).

**Exercise 108.** $\ast : \mathcal{P} \otimes \mathcal{P} \rightarrow \mathcal{P}$ is a morphism of crystals.

That is, the tensor product of two crystals simply consists of the concatenated paths in them.
Remark. We have now defined $B(\lambda)$ explicitly without using $L(\lambda)$. One can prove the Weyl character formula
\[
\text{ch} B(\lambda) = \sum_{w \in W} \det w e^{\omega(\lambda + \rho)} - \rho \prod_{\alpha \in R^+} (1 - e^{-\alpha})
\]
from this, without referring to $L(\lambda)$ or quantum groups. To do this, one builds $\text{ch} L(\lambda)$ and $L(\lambda)$ itself one root at a time, and uses the Demazure character formula.

More specifically: $\forall w \in W$ consider $L_w(\lambda)$, the $\mathfrak{n}^+$-submodule of $L(\lambda)$ generated by $v_w\lambda$, where $v_\lambda$ is the highest weight vector. (Equivalently, it is the submodule generated by the 1-dimensional space $wL(\lambda)$.)

**Theorem 13.2** (Demazure character formula).
\[
\text{ch} L_w(\lambda) = D_w(e^{\lambda}),
\]
where we write $w = s_{i_1} \cdots s_{i_r}$ a reduced (i.e. shortest) decomposition, and set
\[
D_w = D_{s_{i_1}} \cdots D_{s_{i_r}}, \quad D_{s_i}(f) = \frac{f - e^{-\alpha_i}s_i(f)}{1 - e^{-\alpha_i}} = \frac{1}{1 - e^{\alpha_i/2} - e^{-\alpha_i/2}}(fe^{\alpha_i/2} - s_i(fe^{\alpha_i/2})).
\]
That is,
\[
D_{s_i}(e^{\lambda}) = \begin{cases} 
  e^\lambda + e^{\lambda-\alpha_i} + \ldots + e^{s_i\lambda}; & \langle \lambda, \alpha_i^\vee \rangle \geq 0 \\
  0, & \langle \lambda, \alpha_i^\vee \rangle = -1 \\
  -(e^{\lambda+\alpha_i} + \ldots + e^{s_i\lambda-\alpha_i}); & \langle \lambda, \alpha_i^\vee \rangle < -1.
\end{cases}
\]

“I’ve got about six minutes. Is that enough time to tell you about the Langlands program?”

Suppose that $G$ is an algebraic group with Lie algebra $\mathfrak{g}$ which is semisimple. For a given $\mathfrak{g}$, there is the smallest group $G$ above it, $G^{\text{ad}} = G/Z_G$ (whose centre is $\{1\}$). There is also the largest group, $G^{\text{sc}}$ the simply connected cover of $G$, which is still an algebraic group. We have $\pi_1(G^{\text{ad}}) = (P/Q)^*$ and $Z_{G^{\text{sc}}} = P/Q$.

(For example, $SL_n$ is the simply connected group, and $PSL_n$ is the adjoint group.) In this course we have studied the category $\text{Rep} \mathfrak{g} = \text{Rep} G^{\text{sc}}$.

Recall that for $G$ a finite group, the number of irreducible representations is the number of conjugacy classes, but conjugacy classes do not parametrise the representations. For us, however, the representations are parametrised by $P^+ = P/W$.

It is possible, for $G$ an algebraic group, to define $L^G$, the Langlands dual group, where the Lie algebra $\mathfrak{g}$ and the root system $R$ correspond to $\mathfrak{g}^\vee$ and $R^\vee$ (and in fact the torus and the dual torus will swap in the dual group).

If we consider the representations of $G(F)$ where $F = \mathbb{F}_q$ or $F = \mathbb{C}(t)$, these correspond roughly to conjugacy classes of homomorphisms $(W(F) \rightarrow L^G(\mathbb{C}))/L^G(\mathbb{C})$, where $W(F)$ is a “thickening” of the Galois group of $F$. 