FM 5011: Mathematical Background for Finance I
Final exam solutions

1. (15 pts) Consider the binomial market model with two different stocks: \( S_t \) and \( R_t \), \( t = 0, 1, 2, \ldots \). Suppose that \( S_0 = R_0 = 1,000 \). The market interest rate is \( r = 0.06 = \frac{3}{50} \). Stock \( S \) can be multiplied by \( U_S = 1.3 \) or \( D_S = 0.7 \); stock \( R \) can be multiplied by \( U_R = 1.36 \) or \( D_R = 0.76 \).

For this question, your answers should be purely numeric expressions, but you do not have to simplify them. That is, an expression like \( 0.76 \cdot (1300 - 800) / 1.06^2 \) would be acceptable (although this one is not the correct answer for any part of the problem).

(a) What is the arbitrage-free price of a European call option on \( S \) with strike price 1300 and execution time 2?

(b) What is the arbitrage-free price of a European put option on \( R \) with strike price 1300 and execution time 2? Recall that the European put has payoff \( (K - S_T)_+ \).

(c) A forward on \( S \) with expiration 2 and strike price 1300 gives you the right AND OBLIGATION at time 2 to purchase one unit of \( S_2 \) for $1300. You must pay $1300 for the stock even if at time 2 its value is less than $1300. What is the fair price of the forward at time 0?

Solution:

(a) For \( S \), the risk-neutral measure assigns \( P(U_S) = \frac{3}{5} \) and \( P(D_S) = \frac{2}{5} \). Consequently, the price is

\[
\frac{1}{1.06^2} \left( \left( \frac{3}{5} \right)^2 (1690 - 1300)_+ + \left( \frac{2}{5} \right)^2 (910 - 1300)_+ + \left( \frac{2}{5} \right)^2 (490 - 1300)_+ \right) = \frac{1}{1.06^2} \frac{9}{25} (390) = 124.956.
\]

(b) For \( R \), the risk-neutral measure assigns \( P(U_R) = P(D_R) = \frac{1}{2} \). Consequently, the price is

\[
\frac{1}{1.06^2} \left( \frac{1}{4} (1300 - 1.36^2 \cdot 1000)_+ + \frac{1}{2} (1300 - 1.36 \cdot 0.76 \cdot 1000)_+ + \frac{1}{4} (1300 - 0.76^2 \cdot 1000)_+ \right) = 279.281.
\]

(c) The fair price is

\[
\frac{1}{1.06^2} \left( \left( \frac{2}{5} \right)^2 (1690 - 1300)_+ + \frac{3}{5} \cdot \frac{2}{5} (910 - 1300)_+ + \left( \frac{2}{5} \right)^2 (490 - 1300)_+ \right) = -156.995.
\]

An alternative way to see this is to notice that the forward is simply \( S_2 - 1300 \), which means that the replicating portfolio is simply to borrow appropriately discounted 1300 plus one unit of the stock. Consequently, the present value of the contract is \( 1000 - 1300 / 1.06^2 \).

The fair price of a forward is 0 when the strike price is \((1 + r)^T S_0\).

2. (12 pts) For each of the questions below, select the set of correct answers. (It is possible that none of the answers are correct, or that all of them are.) You do not have to justify your answer.

(a) \( X \) is a random variable taking on values 1 through 5, each with positive probability.

i. If \( Y \) is a random variable defined on the same probability space and independent of \( X \), then \( X \) and \( Y \) are uncorrelated. Assume \( Y \) is numeric, so that \( \text{Cov}(X, Y) \) is defined.

ii. If \( Y \) is a random variable defined on the same probability space and independent of \( X \), then \( Y \) must take on the values 1 through 5, each with positive probability.

iii. There exists a random variable \( Y \), measurable with respect to \( X \), which takes on the values 2 through 6, each with positive probability.
iv. There exists a random variable $Y$, whose law is absolutely continuous with respect to the law of $X$, and which takes on the values 2 through 6, each with positive probability.

(b) Let $B_t$ be a Brownian motion, and let $\mathcal{F}_t$ be its natural filtration: $\mathcal{F}_t = \sigma(B_u, u \leq t)$. Let $X_t = B_t^3 - 3tB_t$.

i. $X_t$ is a Gaussian process.

ii. $\mathbb{E}[X_t] = 0$ for all $t$.

iii. $X_1$ and $X_2$ are independent.

iv. There exists an equivalent measure under which $(X_t, t \leq T)$ is a martingale adapted to the filtration $\mathcal{F}$.

(c) Let $S_1(t)$, $S_2(t)$, and $S_3(t)$ be three stocks in the Black-Scholes market.

i. There may be multiple equivalent martingale measures that turn discounted stock prices into martingales.

ii. It is not necessary to know the mean rate of return of $S_1$ to compute the arbitrage-free price of call options on it.

iii. Under any equivalent martingale measure, the three stocks must all have the same mean rate of return.

iv. Under any equivalent martingale measure, the three stocks must all have the same volatility.

Solution:

(a) (i) and (iii) are true, (ii) and (iv) are false. In (iii), take $Y = X + 1$ to see that such a $Y$ really exists.

(b) (ii) and (iv) are true, (i) and (iii) are false. In (i), $B_t^3 - 3tB_t$ is not normally distributed, so it’s not a Gaussian process. In (iv), such a measure definitely exists, because $X_t$ is already a martingale: $dX_t = 3(B_t^2 - t)dB_t$. In (ii), we could check that $X_1$ and $X_2$ are correlated; the easiest way to do this is to write

$$\mathbb{E}[X_1X_2] = \mathbb{E}[\mathbb{E}(X_1X_2|X_1)] = \mathbb{E}[X_1\mathbb{E}(X_2|X_1)] = \mathbb{E}[X_1^2] > 0.$$ 

Here I’ve used the fact that $X$ is a martingale. Realistically, though, there’s just no reason to believe that values of a stochastic process with continuous sample paths would be independent at different times.

(c) (i), (ii), and (iii) are true, (iv) is false. Under the equivalent martingale measure, a.k.a. the risk-neutral measure, all stocks should have mean rate of return equal to the risk-free rate, but they get to keep the volatility they started out with.

3. (12 pts)

(a) Use Itô’s formula to determine whether $X_t = e^{\frac{1}{2}t} \cos B_t$ is a martingale. Show your work.

(b) Using Itô’s isometry or any other method, compute $\text{Var}(X_t)$, where $X_t = t + \int_0^t B_s^2 dB_s$.

Solution:

(a)

$$dX_t = \frac{1}{2} e^{\frac{1}{2}t} \cos B_t dt + e^{\frac{1}{2}t} \left( -\sin(B_t)dB_t - \frac{1}{2} \cos(B_t)dt \right) = -e^{\frac{1}{2}t} \sin(B_t)dB_t.$$ 

Yes, $X_t$ is a martingale.
(b) \[ \text{Var}(t + \int_0^t B_s^2 dB_s) = \text{Var}(\int_0^t B_s^2 dB_s) = \int_0^t \text{Var}(B_s^2) ds = \int_0^t 3s^2 ds = t^3. \]

Here’s an alternative solution:
\[ dX_t = dt + B_t^2 dB_t \implies d\mathbb{E}[X_t] = dt \implies \mathbb{E}[X_t] = t \]
(or you could observe this from the fact that stochastic integrals have mean 0). Also,
\[ d(X_t^2) = 2X_t dt + 2X_t B_t^2 dB_t + B_t^4 dt \implies d\mathbb{E}[X_t^2] = (2 \mathbb{E}[X_t] + \mathbb{E}[B_t^4]) dt = (2t + 3t^2) dt \]
so
\[ \mathbb{E}[X_t^2] = t^2 + t^3, \quad \text{Var}(X_t) = \mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2 = t^3. \]

4. (16 pts) Let \( S_t \) follow a geometric Brownian motion with mean rate of return \( c \) and volatility \( \sigma \); that is,
\[ dS_t = cS_t dt + \sigma S_t dB_t, \]
where \( B_t \) is a Brownian motion. Let \( S_0 = 1 \). Let \( Q \) be the equivalent measure, with respect to which \( e^{-rt} S_t \) is a martingale.

For each of the following processes, write down an expression in terms of \( W_t \), a Brownian motion under \( Q \). Your final answer should not involve integrals, but you do not have to simplify algebraic expressions. That is, an expression like \((e^{W_t} - 3t + 1)^2\) would be acceptable (although this one is not the correct answer for any part of the problem).

(a) \( X_t = B_t \)

(b) \( X_t = \frac{1}{S_t} \) (Hint: start by writing down an expression for \( X_t \) in terms of \( B_t \).)

(c) \( X_t = B_t^2 \)

Solution:

(a) Write
\[ d(e^{-r S_t}) = e^{-r S_t} ((c - r) dt + \sigma dB_t) = e^{-r S_t} \sigma dW_t \]
then solving for \( dW_t \) we get \( dW_t = dB_t + (c - r)/\sigma dt \), and (solving back for \( B_t \) in terms of \( W_t \))
\[ dB_t = dW_t - \frac{c-r}{\sigma} dt \implies B_t = W_t - \frac{c-r}{\sigma} t. \]

(b) Recall that
\[ S_t = e^{(c-\frac{1}{2}\sigma^2)t+\sigma B_t} \implies S_t^{-1} = e^{(-c+\frac{1}{2}\sigma^2)t-\sigma B_t}. \]
Substituting \( B_t \) from part (a), we get
\[ S_t = e^{(-c+\frac{1}{2}\sigma^2)t-\sigma W_t + (c-r)t} = e^{(-r+\frac{1}{2}\sigma^2)t-\sigma W_t}. \]
Of course, we could also arrive at this expression from knowing that under \( W \),
\[ S_t = e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t} \]
and inverting.
(c) Since we solved for $B_t = W_t - (c - r)/\sigma t$ in part (a),

$$B_t^2 = \left( W_t - \frac{c - r}{\sigma} t \right)^2.$$  

5. (15 pts) For random variables $X$ and $Y$ below, compute $E(X|Y)$. Justify your answers.

(a) $X$ and $Y$ are independent standard normal variables (mean 0, variance 1).

(b) $X$ and $Y$ are random variables defined on $[0, 1]$ with the uniform probability measure, whose graphs are given below. (Each axis is to scale, but the scales may be different on the two axes and on different graphs.)

(c) Let $B_t$ be a Brownian motion, and let $X = B_3^2 - 15B_5$ and $Y = B_3$. (Hint: compute $d(B_t^3 - 3tB_t)$, and use the tower law to compare $E(X|Y)$ to $E(X|F_3)$.)

**Solution:**

(a) $E(X|Y) = E[X]$ because $X$ is independent of $Y$, and $E[X] = 0$.

(b) $Y$ takes on three possible values; we don’t actually care what those values are. The three sets of $s$ corresponding to those three values are $(0, 1/4)$, $(1/4, 3/4)$, and $(3/4, 1)$ (it’s not actually clear what happens to the boundaries, and doesn’t matter). The average of $X$ on the first of these sets will be $(0 + 5/4)/2 = 0.625$, the average of $X$ on the second set will be $2.5$, and the average of $X$ on the third of the sets will be $(5 \cdot 3/4 + 5)/2 = 4 + \frac{3}{8} = 4.375$. Consequently,

$$E(X|Y)(s) = \begin{cases} 
0.625, & 0 < s < 1/4 \\
2.5, & 1/4 < s < 3/4 \\
4.375, & 3/4 < s < 1 
\end{cases}$$

(c) We have

$$d(B_t^3 - 3tB_t) = 3B_t^2 dB_t + 3B_t dt - 3tdB_t - 3B_t dt = 3(B_t^2 - t) dB_t,$$

so this process is a martingale. Consequently,

$$E[B_t^3 - 3tB_t|F_s] = B_s^3 - 3sB_s,$$

where $F_s = \sigma(B_u, u \leq s)$. But now, using the tower law,

$$E(Y|X) = E(E(Y|F_3)|X) = E(B_3^3 - 9B_3|X) = B_3^3 - 9B_3,$$

because the result of the inner conditional expectation is $X$-measurable.

Alternatively, you could write

$$B_3^3 - 15B_5 = (B_3 + (B_3 - B_3))^3 - 15B_3 - 15(B_5 - B_3)$$

$$= B_3^3 + 3B_3^2(B_5 - B_3) + 3B_3(B_5 - B_3)^2 + (B_5 - B_3)^3 - 15B_3 - 15(B_5 - B_3).$$
Using the fact that \( B_3 \) is \( X \)-measurable and \( B_5 - B_3 \) is independent of \( X \) and is distributed as \( N(0, 2) \), we conclude
\[
\mathbb{E}(Y|X) = B_3^3 + 0 + 3B_3 \cdot 2 + 0 - 15B_3 - 0 = B_3^3 - 9B_3
\]
as before.

6. (20 pts) The Heston model of stock prices incorporates an uncertain volatility into the stock price. Suppose the stock price \( S_t \) and volatility \( v_t \) follow the SDEs
\[
dS_t = rS_t dt + \sqrt{v_t} S_t dB(t), \quad dv_t = \lambda(\sigma - v_t) dt + c \sqrt{v_t} dW(t)
\]
where \( B \) and \( W \) are independent Brownian motions. Let \( S_0 = v_0 = 1; \) \( c, \lambda, \) and \( \sigma \) are positive constants.

(a) (3 pts) Calculate \( \mathbb{E}[S_t] \). Simplify any integrals in your answer.

(b) (7 pts) Write down the SDE satisfied by \( X_t = S_t v_t \). Use it and your answer to part (a) to write down an ordinary differential equation (ODE) for \( m_t = \mathbb{E}[S_t v_t] \).

(c) (4 pts) Write down the solution of the ODE in part (b), paying attention to the initial conditions. Simplify any integrals in your answer.

**Solution:**

(a) The expectation \( f(t) = \mathbb{E}[S_t] \) satisfies the differential equation \( f'(t) = rf(t) \) with \( f(0) = \mathbb{E}[S_0] = 1 \), so \( \mathbb{E}[S_t] = e^{rt} \).

(b) \[
\begin{align*}
  dX_t &= S_t dv_t + v_t dS_t + dS_t dv_t \\
        &= S_t \lambda \sigma dt - \lambda (S_t v_t) dt + cS_t \sqrt{v_t} dW_t \\
        &\quad + r(S_t v_t) dt + \frac{3}{2} S_t dB_t \\
        &\quad + 0 \quad \text{because } dt^2 = dBdt = dWdt = dBdW = 0.
\end{align*}
\]

(And yes, you should know that \( dBdW = 0 \).) Consequently, \( m_t' = \lambda \sigma \mathbb{E}[S_t] + (r - \lambda) m_t = \lambda \sigma e^{rt} + (r - \lambda) m_t \).

(c) This is a linear ODE. The integrating factor is \( e^{(-r+\lambda)t} \), so
\[
(e^{(-r+\lambda)t} m_t)' = \lambda \sigma e^{rt} e^{(-r+\lambda)t} = \lambda \sigma e^{\lambda t},
\]
or
\[
m_t = e^{(r-\lambda)t} (m_0 + \int_0^t \lambda \sigma e^{\lambda s} ds) = e^{(r-\lambda)t} (1 + \sigma (e^{\lambda t} - 1)).
\]