Review random variables, distribution functions, expectations. Use your favorite probability textbook, or Wikipedia.

1. Let $X$ have the density given by

$$f_X(x) = \begin{cases} 
  x, & 0 \leq x \leq 1; \\
  2 - x, & 1 \leq x \leq 2; \\
  0, & \text{otherwise}.
\end{cases}$$

Verify that $f_X$ is a probability density function, and compute $E[X]$, $E[X^2]$, and $\text{Var}(X) = E[X^2] - E[X]^2$.

2. For each of the three cumulative distribution functions $F_X(x)$ below, compute $P(0 \leq X \leq 1)$. If $X$ has a probability density function, sketch $f_X(x)$. If $X$ has a discrete probability mass function, what is it? Finally, compute $E[X]$. (Assume the last graph is symmetric.)

The rest of the homework concerns the binomial market model.

3. Let $S_0 = 100$, $r = 0.1$, $U = 1.4$, $D = 0.5$.
   
   (a) Draw the binary tree for $T = 4$ time periods.
   
   (b) Consider the contract that pays $9 if $S_4 > 100$, and $0$ if $S_4 < 100$. What is its value at time 0? What are the two possible values at time 1? Pay attention to the power of $(1 + r)$ in your computations, and try to compute this without computing the value of the contract at later times.
   
   (c) Explain in detail how to make money if this option is offered for $3 at time 0. What should you buy or sell at time 2 if $S_1 = 140$ and $S_2 = 70$?

4. As in the lecture, let $S_0 = 80$, $r = 0$, $U = 1.5$, $D = 0.5$. In this exercise, you will derive the no-arbitrage price for the European put option with strike $K = 80$ and expiration $T = 3$. Recall that the buyer of the European put receives the option, but not the obligation, to sell one unit of the underlying stock at time $T$ for $K$. Its payoff at time $T$ is

$$P(K,T) = \begin{cases} 
  K - S_T, & S_T < K; \\
  0, & S_T \geq K.
\end{cases}$$

   (a) Derive the price of the European put at time 0 using risk-neutral valuation.

   (b) Compute the replicating portfolios for all the possible outcomes at time $T = 2$. Compare to the portfolios we computed in lecture for the European call. Do you notice anything interesting?

   (c) Draw a graph to show

$$P(K,T) = K - S_T + C(K,T).$$

   Explain why this gives a way to price $P(K,T)$ relative to $C(K,T)$, and use this call-put parity to check your answer to part (4a).