1. Find the change of measure \( dQ/d\mathbb{P} \) that turns the following processes into martingales on the time interval \([0, T]\). To do that, write down \( dX_t \), and change the definition of Brownian motion in a way that kills the \( dt \) term.
   (a) \( X_t = B_t - 2t \)
   (b) \( X_t = e^{B_t} \)
   (c) \( X_t = e^{B_t + t} \)

Solution:
(a) We have \( dX_t = dB_t - 2dt \), and after the change of measure we want \( dX_t = d\tilde{B}_t \) (i.e. \( X_t = \tilde{B}_t \)). Thus, \( \tilde{B}_t = B_t - 2t \) needs to be a Brownian motion after the change of measure. We apply Girsanov’s theorem:
   \[ \frac{dQ}{d\mathbb{P}}(\omega) = e^{2B_T(\omega) - 2T}, \]
   for some \( T \). Then \( X_t, 0 \leq t \leq T \) will be a martingale under \( Q \).
(b) We have \( dX_t = X_t dB_t + \frac{1}{2} X_t dt = X_t (dB_t + \frac{1}{2} dt) \). After the change of measure, we will have \( dX_t = X_t dB_t \), so \( \tilde{B}_t = dB_t + \frac{1}{2} dt \) or \( \tilde{B}_t = B_t + \frac{1}{2} t \). Girsanov’s theorem tells us that \( \tilde{B}_t \) is a Brownian motion if we use the change of measure
   \[ \frac{dQ}{d\mathbb{P}}(\omega) = e^{-\frac{1}{2} B_T(\omega) - \frac{1}{2} T}, \]
   for some \( T \). Then \( X_t, 0 \leq t \leq T \) will be a martingale under \( Q \).
(c) \( dX_t = X_t dB_t + X_t dt + \frac{1}{2} X_t dt = X_t (dB_t + \frac{3}{2} dt) \). After the change of measure, we will have \( dX_t = X_t \tilde{dB}_t \) with \( \tilde{dB}_t = dB_t + \frac{3}{2} dt \), or \( \tilde{B}_t = B_t + \frac{3}{2} t \). Girsanov’s theorem gives the change of measure
   \[ \frac{dQ}{d\mathbb{P}}(\omega) = e^{-\frac{3}{2} B_T(\omega) - \frac{3}{2} T}. \]

2. Compute
   \[ d \left( (B_t + qt)^2 - t \right) (e^{-q B_t - \frac{1}{2} q^2 t}) \]
   and finish our proof from lecture that \( \tilde{B}_t \) is a Brownian motion under the measure \( Q \) given in Girsanov’s theorem.

Solution: Note that we only care about the \( dt \) terms cancelling. Note also that \( e^{-q B_t - \frac{1}{2} q^2 t} \) is known to be a martingale, with
   \[ d(e^{-q B_t - \frac{1}{2} q^2 t}) = -qe^{-q B_t - \frac{1}{2} q^2 t} dB_t. \]

This means
\[
\begin{align*}
&d \left( (B_t + qt)^2 - t \right) \left( e^{-q B_t - \frac{1}{2} q^2 t} \right) \\
&= ((B_t + qt)^2 - t) \cdot \left( -qe^{-q B_t - \frac{1}{2} q^2 t} \right) dB_t + (2q(B_t + qt) - 1) dt + 2(B_t + qt) \cdot dB_t + dt \left( e^{-q B_t - \frac{1}{2} q^2 t} \right) \\
&\quad + 2(B_t + qt) \left( -qe^{-q B_t - \frac{1}{2} q^2 t} \right) dt.
\end{align*}
\]

The \( dt \) terms do in fact happily cancel.
3. Use the Black–Scholes pricing formula to compute the (current, time-0) price of a European call in a market with $S_0 = 100$, $r = 0.01$, strike $K = 100$, and maturity $T = 3.7$. Obtain a numeric answer.

Recall the call–put parity: a put option has pay-off $(K - S_T)_+$ (it allows you to *sell* one unit of stock for $\$K$). Let $C_t(K,T)$ be the price of a call option with strike $K$ and maturity $T$ at time $t$, and let $P_t(K,T)$ be the same for a put option. Because $(S_T - K_+)_+ - (K - S_T)_+ = S_T - K$, we have $C_T(K,T) - P_T(K,T) = S_T - K$.

Recall that under the measure $Q$, discounted portfolio prices are martingales. Take conditional expectations (don’t forget the discounting!) to derive the relationship between fair prices $C_t(K,T)$ and $P_t(K,T)$ at time $t \leq T$.

Finally use the relationship you derived to obtain a numeric value of the put option with the same data as you had for the call option above there. Check that your answer makes sense (in particular, should the call or the put be more expensive?).

**Solution:** The Black-Scholes pricing formula for a call (i.e. terminal pay-off $(S_T - K)_+$) gives

$$V(T-t, S_t) = S_t \Phi \left( \frac{\ln(S_t/K) + (r + \frac{1}{2} \sigma^2(T-t))}{\sigma \sqrt{T-t}} \right) - K e^{-rT} \Phi \left( \frac{\ln(S_t/K) + (r + \frac{1}{2} \sigma^2(T-t))}{\sigma \sqrt{T-t}} - \sigma \sqrt{T-t} \right)$$

Putting in $T-t = 3.7$, $S_t = 100$, $K = 100$, and $r = 0.01$, we notice that I utterly failed to give a value for $\sigma$. Let’s take $\sigma = 1$.

Then

$$V(3.7, 100) = 66.99, \quad \sigma = 1.$$  

If we take $\sigma = 0.001$, then $V(3.7, 100) = 3.63$. Notice that when the volatility is tiny, stock price is essentially deterministic, and we can treat its mean rate of return as equal to $r$. Consequently, for $\sigma \approx 0$ the price of the option should be $S_0 - Ke^{-rT} \approx 3.63$, exactly as we see here.

For call–put parity, take conditional expectations of $C_T(K, T) - P_T(K, T) = K - S_T$ under the risk-neutral measure:

$$\mathbb{E}(C_T(K, T)|\mathcal{F}_t) - \mathbb{E}(P_T(K, T)|\mathcal{F}_t) = \mathbb{E}(S_T|\mathcal{F}_t) - K$$

or, multiplying everything by $e^{r(T-t)}$,

$$C_t(K, T) - P_t(K, T) = S_t - e^{-r(T-t)} K.$$

In particular, in our problem we have

$$C_0(K, T) - P_0(K, T) = S_0 - e^{-rT} K$$

so

$$P_0(K, T) = C_0(K, T) + e^{-rT} K - S_0 = C_0(K, T) - 3.63 \approx 63.37, \quad \sigma = 1$$

or $\approx 0$ for $\sigma = 0.001$. Notice that $P_0 < C_0$ since $K < S_0e^{rt}$, i.e. it’s likely that the terminal stock price will be higher than $K$; and as $\sigma \to 0$ here, $P_0 \to 0$. (If we had $K > S_0e^{rt}$ then we would instead have $S_0 \to 0$ as $\sigma \to 0$.)
