1. (Cox – Ingersoll – Ross model) This is another model of interest rates. It shares the mean-reverting property of the Vasicek interest rate model, but it also has the feature that interest rates cannot become negative. (They are not normally distributed.)

Consider the process $r_t$ satisfying

$$dr_t = a(b - r_t)dt + \sigma \sqrt{r_t} dB_t, \quad r(0) = r_0.$$ 

This is called the CIR process, or the square root process. Notice that this doesn’t satisfy the conditions of the theorem on the existence and uniqueness of strong solutions! Nonetheless, it’s possible to show that there is a unique strong solution.

Write down the ODE satisfied by $m_t = \mathbb{E}[r_t]$. (You can do this despite the $\sqrt{r_t}$ term.) Solve this ODE to compute $m_t$.

Write down the ODE satisfied by $q_t = \mathbb{E}[r_t^2]$. (You can do this despite the $\sqrt{T}$ term.) Solve this ODE to compute $q_t$. Thus find $\text{Var}[r_t]$.

You should be getting $m_t = b + e^{-at}(r_0 - b)$ and

$$q_t = r_0^2 e^{-2at} + (2ab + a^2) \left( \frac{b}{2a}(1 - e^{-2at}) + \frac{r_0 - b}{a}(1 - e^{-at})e^{-at} \right).$$

2. In this problem we look at a system of SDEs, and try to find the mean and variance of the solutions.

(a) Consider the system of SDEs

$$\begin{aligned}
    dS_1(t) &= c_1 S_1(t) dt + \sigma_{11} S_1(t) dB_1(t) + \sigma_{12} S_2(t) dB_2(t) \\
    dS_2(t) &= c_2 S_2(t) dt + \sigma_{21} S_1(t) dB_1(t) + \sigma_{22} S_2(t) dB_2(t).
\end{aligned}$$

Let $m_1(t) = \mathbb{E}[S_1(t)]$, $m_2(t) = \mathbb{E}[S_2(t)]$. Write down and solve the ODEs satisfied by $m_1$ and $m_2$.

(b) Consider the system of SDEs

$$\begin{aligned}
    dS_1(t) &= c_1 S_2(t) dt + \sigma_{11} S_1(t) dB_1(t) + \sigma_{12} S_1(t) dB_2(t) \\
    dS_2(t) &= c_2 S_1(t) dt + \sigma_{22} S_2(t) dB_2(t).
\end{aligned}$$

Write down the system of ODEs satisfied by $m_1(t) = \mathbb{E}[S_1(t)]$ and $m_2(t) = \mathbb{E}[S_2(t)]$.

(c) Find two linear combinations $m_1(t) + km_2(t)$, $m_1(t) + \tilde{k}m_2(t)$ for which

$$\begin{aligned}
    d(m_1(t) + km_2(t)) &= \lambda(m_1(t) + km_2(t)) dt \\
    d(m_1(t) + \tilde{k}m_2(t)) &= \tilde{\lambda}(m_1(t) + \tilde{k}m_2(t)) dt
\end{aligned}$$

Use this to solve the system of ODEs for $m_1$ and $m_2$.

3. Consider the multidimensional market model,

$$dS_i(t) = c_i(t) S_i(t) dt + S_i(t) \sum_{i=1}^{n} \sigma_{ij}(t) dB_j(t).$$

Define the processes $W_i(t)$ as follows:

$$dW_i(t) = \sum_{j=1}^{n} \frac{\sigma_{ij}(t)}{\sqrt{\sum_{j=1}^{n} \sigma_{ij}^2(t)}} dB_j(t).$$
(a) The processes $W_i(t)$ are continuous martingales. Show that $(dW_i(t))^2 = dt$ for all $i$; this will show that $W_i(t)$ are all Brownian motions. (Recall that $(dB_j)^2 = dt$ and $dB_j dB_k = 0$ for $j \neq k$.)

(b) Rewrite $dS_i(t)$ in terms of $W_i(t)$. This shows that the $S_i(t)$ individually behave like stock prices in the usual Black–Scholes market.

(c) The processes $W_i(t)$ are not independent. Compute $d(W_i(t) W_j(t))$. Use your answer to find Cov($W_i(t), W_j(t)$).

4. **This problem best done on a computer, or at least with many digits of precision.**

   Consider the binomial model. Let $C_t(K,T)$ be the price at time $t$ of a European call option with strike $K$ and maturity $T$. Work this out for all $t$ when $S_0 = 10$, $T = 3$, $U = 1.4$, $D = 0.7$, $r = 0.1$, and $K = 11$. (Hint: risk-neutral valuation is the fast way of doing this.)

   Now consider the following exotic option: at time $\tau = 2$ you will have the option (but not the obligation) of buying one of the above European calls with strike $K$ and maturity $T$, for a price of $\kappa = 3$. That is, this is an option on the European call with strike $\kappa$ and maturity $\tau$. Work out the fair price of this exotic option. (Hint: at time $\tau$ your pay-off is $(C_\tau(K,T) - \kappa)_+$.)

   What does call–put parity look like for the exotic option? Use call–put parity to work out the price of the corresponding exotic put (which gives you the option of selling one European call with strike $K$ and maturity $T$ at time $\tau$ for a price $\kappa$).

5. **This problem best done on a computer, or at least with many digits of precision.**

   Consider the binomial market, but now with two stocks. The interest rate is $r = 0.1$, the stock prices are $S_1(0) = 10$ and $S_2(0) = 15$, and the possibilities for the two stocks are $U_1 = 1.4$, $D_1 = 0.8$ and $U_2 = 1.2$, $D_2 = 0.7$. Find the fair price of a the European call on $S_1 + S_2$, with strike $K = 27$ and maturity $T = 3$. (That is, the pay-out of the contract is is $(S_1(T) + S_2(T) - K)_+$.)

   What is the replicating portfolio at time $T = 2$, if $S_1(2) = 28.9$ and $S_2(2) = 12.6$?