FM 5011: Mathematical Background for Finance I
Homework 11 solution

1. (Cox – Ingersoll – Ross model) This is another model of interest rates. It shares the mean-reverting property of the Vasicek interest rate model, but it also has the feature that interest rates cannot become negative. (They are not normally distributed.)

Consider the process $r_t$ satisfying

$$dr_t = a(b - r_t)dt + \sigma \sqrt{r_t}dB_t, \quad r(0) = r_0.$$ 

This is called the CIR process, or the square root process. Notice that this doesn’t satisfy the conditions of the theorem on the existence and uniqueness of strong solutions! Nonetheless, it’s possible to show that there is a unique strong solution.

Write down the ODE satisfied by $m_t = \mathbb{E}[r_t]$. (You can do this despite the $\sqrt{r_t}$ term.) Solve this ODE to compute $m_t$.

Write down the ODE satisfied by $q_t = \mathbb{E}[r_t^2]$. (You can do this despite the $\sqrt{T}$ term.) Solve this ODE to compute $q_t$. Thus find $\text{Var}[r_t]$.

You should be getting $m_t = b + e^{-at}(r_0 - b)$ and

$$q_t = r_0^2e^{-2at} + (2ab + \sigma^2)\left(\frac{b}{2a}(1 - e^{-2at}) + \frac{r_0 - b}{a}(1 - e^{-at})e^{-at}\right).$$ 

Solution:

$$dm_t = a(b - m_t)dt;$$ 

notice that we don’t care about the $dB$ term. Solving this equation,

$$d(b - m_t) = -a(b - m_t)dt \implies b - m_t = Ce^{-at} \implies m_t = b - Ce^{-at} = b - (b - r_0)e^{-at}.$$ 

You could unleash the heavy machinery of linear ODEs on the equation, of course, but it’s simpler to find a change of variables $u_t = b - m_t$ that makes the equation easy to solve. Notice that $m_0 = \mathbb{E}[r_0] = r_0$.

To compute $q_t$, write

$$d(r_t^2) = 2r_t dr_t + (dr_t)^2 = (-2ar_t^2 + 2abr_t + \frac{1}{2}\sigma^2 r_t)dt + 2\sigma r_t^{3/2}dB_t,$$

so

$$q'(t) = -2aq(t) + (2ab + \frac{1}{2}\sigma^2)m(t) = -2aq(t) + (2ab + \frac{1}{2}\sigma^2)(b + e^{-at}(r_0 - b)).$$ 

Premultiplying by $e^{2at}$ gives

$$\frac{d}{dt}(e^{2at}q(t)) = e^{2at}(2ab + \frac{1}{2}\sigma^2)(b + e^{-at}(r_0 - b)) = (2ab^2 + \frac{1}{2}\sigma^2 b)e^{2at} + (2ab + \frac{1}{2}\sigma^2)(r_0 - b)e^{at}$$

so

$$q(t) = e^{-2at}\left(r_0^2 + \frac{2ab^2 + \frac{1}{2}\sigma^2 b}{2a}(e^{2at} - 1) + \frac{(2ab + \frac{1}{2}\sigma^2)(r_0 - b)}{a}(e^{at} - 1)\right).$$ 

Simplifying should get us to the right answer. The variance is $\text{Var}(r_t) = q_t - m_t^2$; it will consist of $\frac{b\sigma^2}{2a}$ (the steady-state variance) plus exponentially decaying terms (i.e. terms multiplied by $e^{-at}$ and $e^{-2at}$).

2. In this problem we look at a system of SDEs, and try to find the mean and variance of the solutions.
(a) Consider the system of SDEs
\[
\begin{align*}
    dS_1(t) &= c_1 S_1(t) dt + \sigma_{11} S_1(t) dB_1(t) + \sigma_{12} S_2(t) dB_2(t) \\
    dS_2(t) &= c_2 S_2(t) dt + \sigma_{22} S_2(t) dB_2(t).
\end{align*}
\]
Let \( m_1(t) = \mathbb{E}[S_1(t)] \), \( m_2(t) = \mathbb{E}[S_2(t)] \). Write down and solve the ODEs satisfied by \( m_1 \) and \( m_2 \).

(b) Consider the system of SDEs
\[
\begin{align*}
    dS_1(t) &= c_1 S_1(t) dt + \sigma_{11} S_1(t) dB_1(t) + \sigma_{12} S_2(t) dB_2(t) \\
    dS_2(t) &= c_2 S_2(t) dt + \sigma_{22} S_2(t) dB_2(t).
\end{align*}
\]
Write down the system of ODEs satisfied by \( m_1(t) = \mathbb{E}[S_1(t)] \) and \( m_2(t) = \mathbb{E}[S_2(t)] \).

(c) Find two linear combinations \( m_1(t) + km_2(t) \), \( m_1(t) + \tilde{k}m_2(t) \) for which
\[
\begin{align*}
    dm_1(t) + km_2(t) &= \lambda(m_1(t) + km_2(t)) dt \\
    dm_1(t) + \tilde{k}m_2(t) &= \hat{\lambda}(m_1(t) + \tilde{k}m_2(t)) dt.
\end{align*}
\]
Use this to solve the system of ODEs for \( m_1 \) and \( m_2 \).

**Solution:**

(a) Expectations of all Brownian motions are zero, so
\[
\begin{align*}
    \left\{ m'_1(t) = c_1 m_2(t) m'_2(t) = c_2 m_1(t) \right\}
\end{align*}
\]

(b) Setting
\[
\begin{align*}
    (m_1(t) + km_2(t))' &= c_1 m_2(t) + kc_2 m_1(t),
\end{align*}
\]
we need to solve
\[
\begin{align*}
    \lambda = kc_2, \quad \lambda k = c_1 \implies k^2 = \frac{c_1}{c_2},
\end{align*}
\]
which means \( k = \pm c_1/c_2 \) and correspondingly \( \lambda = \pm c_1 \) (with the same signs). That is,
\[
\begin{align*}
    \left\{ \left( m_1(t) + \frac{c_1}{c_2} m_2(t) \right)' = c_2 \left( m_1(t) + \frac{c_1}{c_2} m_2(t) \right) dt \right\} \left( m_1(t) - \frac{c_1}{c_2} m_2(t) \right)' = -c_2 \left( m_1(t) - \frac{c_1}{c_2} m_2(t) \right) dt
\end{align*}
\]
This tells us
\[
\begin{align*}
    m_1(t) + \frac{c_1}{c_2} m_2(t) = K_1 \exp(c_2 t), \quad K_1 = S_1(0) + \frac{c_1}{c_2} S_2(0)
\end{align*}
\]
and
\[
\begin{align*}
    m_1(t) - \frac{c_1}{c_2} m_2(t) = K_2 \exp(-c_2 t), \quad K_2 = S_1(0) - \frac{c_1}{c_2} S_2(0).
\end{align*}
\]
Consequently,
\[
\begin{align*}
    m_1(t) &= \frac{1}{2} K_1 e^{c_2 t} - \frac{1}{2} K_2 e^{-c_2 t}, \\
    m_2(t) &= \frac{1}{2} \frac{c_2}{c_1} (K_1 e^{c_2 t} - K_2 e^{-c_2 t}).
\end{align*}
\]

3. Consider the multidimensional market model,
\[
\begin{align*}
    dS_i(t) &= c_i(t) S_i(t) dt + S_i(t) \sum_{i=1}^{n} \sigma_{ij}(t) dB_j(t).
\end{align*}
\]
Define the processes \( W_i(t) \) as follows:
\[
\begin{align*}
    dW_i(t) &= \sum_{j=1}^{n} \frac{\sigma_{ij}(t)}{\sqrt{\sum_{j=1}^{n} \sigma_{ij}^2(t)}} dB_j(t).
\end{align*}
\]
(a) The processes $W_i(t)$ are continuous martingales. Show that $(dW_i(t))^2 = dt$ for all $i$; this will show that $W_i(t)$ are all Brownian motions.

(b) Rewrote $dS_i(t)$ in terms of $W_i(t)$. This shows that the $S_i(t)$ individually behave like stock prices in the usual Black–Scholes market.

(c) The processes $W_i(t)$ are not independent. Compute $d(W_i(t)W_j(t))$. Use your answer to find Cov($W_i(t), W_j(t)$).

**Solution:**

(a) Because $dB_i dB_j = 0$,

$$(dW_i(t))^2 = \sum_{j=1}^{n} \frac{\sigma_{ij}^2(t)}{\sum_{j=1}^{n} \sigma_{ij}^2(t)} (dB_j(t))^2 = dt.$$  

(b) The $dB$ terms differ only by the factor of the denominator, which is the same for all $j$, so

$$dS_i(t) = c_i(t) S_i(t) dt + S_i(t) \sqrt{\sum_{j=1}^{n} \sigma_{ij}^2(t)} dW_i(t).$$

(c) Using product rule,

$$d(W_i(t)W_j(t)) = dW_i(t)W_j(t) + dW_j(t)W_i(t) + dW_i(t)dW_j(t).$$

Since we’ll be interested in $E[W_i(t)W_j(t)]$ for the covariance, we will ignore the first two terms, which are purely $dB$ and don’t contribute to the expectation. The third term is

$$dW_i(t)dW_j(t) = \sum_{k=1}^{n} \frac{\sigma_{ik}(t)\sigma_{jk}(t)}{\sqrt{(\sum_{k=1}^{n} \sigma_{ik}^2(t)) (\sum_{j=1}^{n} \sigma_{jk}^2(t))}} (dB_k(t))^2$$

$$= \left( \sum_{k=1}^{n} \frac{\sigma_{ik}(t)\sigma_{jk}(t)}{\sqrt{(\sum_{k=1}^{n} \sigma_{ik}^2(t)) (\sum_{j=1}^{n} \sigma_{jk}^2(t))}} \right) dt.$$  

Consequently,

$$\text{Cov}(W_i(t), W_j(t)) = E[W_i(t)W_j(t)] = \int_{0}^{t} \left( \sum_{k=1}^{n} \frac{\sigma_{ik}(u)\sigma_{jk}(u)}{\sqrt{(\sum_{k=1}^{n} \sigma_{ik}^2(u)) (\sum_{j=1}^{n} \sigma_{jk}^2(u))}} \right) du.$$  

4. **This problem best done on a computer, or at least with many digits of precision.**

Consider the binomial model. Let $C_t(K, T)$ be the price at time $t$ of a European call option with strike $K$ and maturity $T$. Work this out for all $t$ when $S_0 = 10$, $T = 3$, $U = 1.4$, $D = 0.7$, $r = 0.1$, and $K = 11$.

(Hint: risk-neutral valuation is the fast way of doing this.)

Now consider the following exotic option: at time $\tau = 2$ you will have the option (but not the obligation) of buying one of the above European calls with strike $K$ and maturity $T$, for a price of $\kappa = 3$. That is, this is an option on the European call with strike $\kappa$ and maturity $\tau$. Work out the fair price of this exotic option. (Hint: at time $\tau$ your pay-off is $(C_t(K, T) - \kappa)_+).$

What does call–put parity look like for the exotic option? Use call–put parity to work out the price of the corresponding exotic put (which gives you the option of selling one European call with strike $K$ and maturity $T$ at time $\tau$ for a price $\kappa$).

**Solution:** The prices of the call are given in the table below. The risk-neutral measure is to go up with probability $4/7$ and down with probability $3/7$. Recall that the possible pay-offs are $(S_0U^kD^{3-k} - K)_+$, which is positive for the top two outcomes and zero for the other two.
\begin{tabular}{|c|c|c|}
\hline
\textit{t} & \textit{S}_t & \textit{C}_t(K,T) \\
\hline
0 & 10 & 2.310293 \\
1 & 14 & 4.520661 \\
1 & 7 & 0.5619835 \\
2 & 19.6 & 8.709091 \\
2 & 9.8 & 1.236364 \\
2 & 4.9 & 0 \\
3 & 27.44 & 16.44 \\
3 & 13.72 & 2.72 \\
3 & 6.86 & 0 \\
3 & 3.43 & 0 \\
\hline
\end{tabular}

This means that my pay-off at \( \tau = 2 \) is 5.709091 if \( S_2 = 19.6 \), and 0 otherwise. Consequently, my option is worth \( \frac{1}{4} \cdot 5.709091/1.1^2 = 1.179564 \).

The call-put parity says \((C_t - \kappa)_+ - (C_t - \kappa)_- = C_t - \kappa\). Consequently, the price of the exotic put is

\[
\text{put} = \text{call} - \frac{1}{1.1^2} (C_0 - \kappa) = 1.179564 - \frac{1}{1.1^2} (2.310293 - 3) = 1.74957.
\]

5. This problem best done on a computer, or at least with many digits of precision.

Consider the binomial market, but now with two stocks. The interest rate is \( r = 0.1 \), the stock prices are \( S_1(0) = 10 \) and \( S_2(0) = 15 \), and the possibilities for the two stocks are \( U_1 = 1.4 \), \( D_1 = 0.8 \) and \( U_2 = 1.2 \), \( D_2 = 0.7 \). Find the fair price of a the European call on \( S_1 + S_2 \), with strike \( K = 27 \) and maturity \( T = 3 \). (That is, the pay-out of the contract is \((S_1(T) + S_2(T) - K)_+\))

What is the replicating portfolio at time \( T = 2 \), if \( S_1(2) = 28.9 \) and \( S_2(2) = 12.6 \)?

**Solution:** We start by finding the equivalent martingale measure. There’s been some confusion over whether the stocks are moving independently, and the answer is, I don’t care. As long as all four combinations are possible, I can change measures so that the stocks move independently.

The equivalent martingale measure assigns probability \( \frac{1}{2} \) each to the first stock moving up and down, and probability \( \frac{4}{5} \) to the second stock moving up (and \( \frac{1}{5} \) to the second stock moving down). For any correlation structure for these moves, the stocks are now martingales. This suggests that there won’t be a unique martingale measure in this problem!

Let’s assume that we want the stocks to be independent. Then the value is

\[
\frac{1}{1.1^3} \left( \frac{1}{2} (\frac{4}{5})^3 \Phi(S_1(0)U_1^3, S_2(0)U_2^3) + \ldots \right) = 0.9190684.
\]

(You really want to do this on a computer, rather than looking at all 16 outcomes by hand!)

The stocks can’t be perfectly correlated (one goes up with probability \( \frac{1}{2} \), the other goes up with probability \( \frac{4}{5} \)), but we could look at the scenario where if stock 1 goes up, stock 2 is very likely to go up also. That is, if stock 1 goes up then stock 2 goes up, and if stock 1 goes down then stock 2 goes up with probability \( \frac{3}{5} \) and down with probability \( \frac{2}{5} \). Then the value we get is

\[
\frac{1}{1.1^3} \left( \frac{1}{2} (\frac{3}{5})^3 \Phi(S_1(0)U_1^3, S_2(0)U_2^3) + \ldots \right) = 3.789391
\]

We could also compute the most anti-correlated case, where if \( S_1 \) goes up, \( S_2 \) goes up with probability that’s as small as possible, namely \( \frac{3}{5} \), and if \( S_1 \) goes down then \( S_2 \) goes up. In that case the value we get is

\[
\frac{1}{1.1^3} \left( \frac{1}{2} (\frac{3}{5})^3 \Phi(S_1(0)U_1^3, S_2(0)U_2^3) + \ldots \right) = 1.851781.
\]
We see that the independent case actually gives a lower price. It’s not very easy to predict what the smallest possible price we can get, but these three scenarios seem like natural choices to examine and get a feel for how much the price might vary.

We now look at the replicating portfolio: we would like

\[
\begin{align*}
  a \cdot 40.46 + b \cdot 15.12 + c \cdot 1.1 &= 28.58 \\
  a \cdot 23.12 + b \cdot 15.12 + c \cdot 1.1 &= 11.24 \\
  a \cdot 40.46 + b \cdot 8.82 + c \cdot 1.1 &= 22.28 \\
  a \cdot 23.12 + b \cdot 8.82 + c \cdot 1.1 &= 4.94
\end{align*}
\]

As we see, this has four equations in three unknowns, so we can’t solve it: there isn’t any replicating portfolio in this case! This agrees with the fact that we found multiple possible prices using the equivalent martingale measure method.