1. Let $X$ have the density given by

$$f_X(x) = \begin{cases} 
  x, & 0 \leq x \leq 1; \\
  2 - x, & 1 \leq x \leq 2; \\
  0, & \text{otherwise}.
\end{cases}$$

Verify that $f_X$ is a probability density function, and compute $E[X]$, $E[X^2]$, and $\text{Var}(X) = E[X^2] - E[X]^2$.

**Solution:** To check that $f_X$ is a probability density function, we need to verify that $f_X \geq 0$ always (true), and $
\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ (also true, since $\int_0^1 x\,dx + \int_1^2 (2 - x)\,dx = 1$).

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 x(2 - x)\,dx = 1.$$ (We could also get this by symmetry: the density $f_X$ is symmetric around 1, so the average value of $X$ is 1.)

$$E[X^2] = \int_0^1 x^3\,dx + \int_1^2 x^2(2 - x)\,dx = \frac{7}{6}.$$ Therefore,

$$\text{Var}(X) = \frac{7}{6} - 1 = \frac{1}{6}.$$ 

2. For each of the three cumulative distribution functions $F_X(x)$ below, compute $P(0 \leq X \leq 1)$. If $X$ has a probability density function, sketch $f_X(x)$. If $X$ has a discrete probability mass function, what is it? Finally, compute $E[X]$ (Assume the last graph is symmetric.)

**Solution:**

$$P(0 \leq X \leq 1) = P(X \leq 1) - P(X < 0) = F_X(1) - \lim_{\epsilon \to 0} F_X(-\epsilon).$$ Consequently:

For the first distribution, this is 1/3. (The line has slope 1/3.)

For the second distribution, this is 1/8. Note that it’s not 0! $X$ has a positive probability of being equal to 0.

For the third distribution, this is 0.5 - 0.1 = 0.4.

We can compute expectations from the density, but it’s easier to do it by symmetry:

The first distribution is rotationally symmetric about (1/2, 1/2). Thus, its density will be symmetric about 1/2, and $E[X] = 1/2$. We can also compute $E[X] = \int_{1/2}^{3/2} x\frac{1}{2}\,dx = \frac{1}{2}$.

The second distribution does not have a density. Instead, it has a probability mass function. We have

$$P(X = 0) = F_X(0) - F_X(-\epsilon) = \frac{1}{8}; \quad P(X = 1.5) = F_X(1.5) - F_X(1.5 - \epsilon) = \frac{1}{8};$$

$$P(X = 2) = F_X(2) - F_X(2 - \epsilon) = \frac{1}{4}; \quad P(X = 3) = F_X(3) - F_X(3 - \epsilon) = \frac{1}{4};$$

$$P(X = 4) = F_X(4) - F_X(4 - \epsilon) = \frac{1}{4}.$$
Hence,

\[ E[X] = \frac{1}{8} \cdot 0 + \frac{1}{8} \cdot 1.5 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 3 + \frac{1}{4} \cdot 4 = \frac{39}{16} = 2.4375. \]

For the third distribution, you can’t actually compute the density, because I didn’t give you a formula for \( F_X \). But, because \( F_X \) is rotationally symmetric about \((1, 1/2)\), we know the density will be symmetric about 1. (And because \( F_X \) is concave up to the left of 1, and concave down to the right of 1, we know that the density is increasing to the left of 1 and decreasing to the right of 1, with a maximum at 1.) This means that \( E[X] = 1 \).

Sketches of densities below.

For the rest of the homework concerns the binomial market model.

3. Let \( S_0 = 100 \), \( r = 0.1 \), \( U = 1.4 \), \( D = 0.5 \).

(a) Draw the binary tree for \( T = 4 \) time periods.

(b) Consider the contract that pays \$9 if \( S_4 > 100 \), and \$0 if \( S_4 < 100 \). What is its value at time 0? What are the two possible values at time 1? Pay attention to the power of \((1 + r)\) in your computations, and try to compute this without computing the value of the contract at later times.

(c) Explain in detail how to make money if this option is offered for \$3 at time 0. Draw out all the possibilities, and solve for the portfolios you should hold at all the time periods. What should you buy or sell at time 2 if \( S_1 = 140 \) and \( S_2 = 70 \)?

Solution: Binary tree:

The two dashed outlines will be relevant for computing the value at time 1.

The risk-neutral measure is given by \( p = 2/3 \): then \( pU + (1 - p)D = (1 + r) \).
The contract pays $9 in the top two outcomes, and otherwise 0. Thus, the value at time 0 is:

\[ V_0 = \frac{1}{(1 + r)^4} \left( p^4 \cdot 9 + 4p^3(1 - p) \cdot 9 \right) \approx 3.6427 \]

At time 1, I see a three-period tree: either the top four outcomes (if stock went up) or the bottom four outcomes (if stock went down). These two three-period trees are circled on the big tree. I then compute the value exactly as above, for this smaller tree:

\[ [V_1|S_1 = 140] = \frac{1}{(1 + r)^3} \left( p^3 \cdot 9 + 3p^2(1 - p) \cdot 9 \right) \approx 5.0088 \]

\[ [V_1|S_1 = 50] = \frac{1}{(1 + r)^3} \left( p^3 \cdot 9 \right) \approx 2.0035 \]

While we’re at it, let’s solve for the rest of the values:

\[ [V_3|S_3 = 274.4] = \frac{1}{(1 + r)^2} \cdot 9 \approx 8.1818 \]

\[ [V_3|S_3 = 98] = \frac{1}{(1 + r)} (p \cdot 9) \approx 5.4545 \]

\[ [V_3|S_3 = 35] = 0 \]

\[ [V_3|S_3 = 12.5] = 0 \]

\[ [V_2|S_2 = 196] = \frac{1}{(1 + r)^2} \left( p^2 \cdot 9 + 2p(1 - p) \cdot 9 \right) \approx 6.6116 \]

\[ [V_2|S_2 = 70] = \frac{1}{(1 + r)^2} \left( p^2 \cdot 9 \right) \approx 3.3058 \]

\[ [V_2|S_2 = 25] = 0 \]

There are 10 replicating portfolios to solve for. The systems of equations with solutions (and values) are listed below.

\( T = 3: \)
\[
\begin{align*}
1.1x + 384.16y &= 9 \\
1.1x + 137.2y &= 9 \\
1.1x + 137.2y &= 9 \\
1.1x + 49y &= 0 \\
1.1x + 49y &= 0 \\
1.1x + 17.5y &= 0 \\
1.1x + 6.25y &= 0
\end{align*}
\]

\[ \implies y = 0, \ x \approx 8.1818; x + 274.4y \approx 8.1818 \]

\[
\begin{align*}
1.1x + 137.2y &= 9 \\
1.1x + 49y &= 0 \\
1.1x + 17.5y &= 0 \\
1.1x + 6.25y &= 0
\end{align*}
\]

\[ \implies x \approx -4.54545, \ y \approx 0.1020; x + 98y \approx 5.4545 \]

\[
\begin{align*}
1.1x + 49y &= 0 \\
1.1x + 17.5y &= 0 \\
1.1x + 6.25y &= 0
\end{align*}
\]

\[ \implies x = y = 0, x + 35y = 0 \]

\[
\begin{align*}
1.1x + 17.5y &= 0 \\
1.1x + 6.25y &= 0
\end{align*}
\]

\[ \implies x = y = 0, x + 12.5y = 0 \]

\( T = 2: \)
\[
\begin{align*}
1.1x + 274.4y &= 8.1818 \\
1.1x + 98y &= 5.4545 \\
1.1x + 98y &= 5.4545 \\
1.1x + 35y &= 0 \\
1.1x + 35y &= 0 \\
1.1x + 12.5y &= 0
\end{align*}
\]

\[ \implies x \approx 3.5813, \ y \approx 0.01546; x + 196y \approx 6.6116 \]

\[
\begin{align*}
1.1x + 98y &= 5.4545 \\
1.1x + 35y &= 0 \\
1.1x + 35y &= 0 \\
1.1x + 12.5y &= 0
\end{align*}
\]

\[ \implies x \approx -2.7548, \ y \approx 0.0866; x + 70y \approx 3.3058 \]

\[
\begin{align*}
1.1x + 35y &= 0 \\
1.1x + 12.5y &= 0
\end{align*}
\]

\[ \implies x = y = 0, x + 25y = 0 \]
\[ T = 1: \]
\[
\begin{align*}
1.1x + 196y &= 6.6116 \\
1.1x + 70y &= 3.3058 \\
1.1x + 70y &= 3.3058 \\
1.1x + 25y &= 0
\end{align*}
\]
\[ \Rightarrow x \approx 1.3357, \; y \approx 0.0262; \; x + 140y \approx 5.0088 \]
\[
\begin{align*}
1.1x + 70y &= 3.3058 \\
1.1x + 140y &= 5.0088 \\
1.1x + 50y &= 2.0035
\end{align*}
\]
\[ \Rightarrow x \approx -1.6696, \; y \approx 0.0735; \; x + 50y \approx 2.0035 \]
\[ T = 0: \]
\[
\begin{align*}
1.1x + 140y &= 5.0088 \\
1.1x + 50y &= 2.0035
\end{align*}
\]
\[ \Rightarrow x \approx 0.30356, \; y \approx 0.03339, \; x + 100y \approx 3.6427 \]

If the option is traded for $3, then since $3 is less than the value of the option, you should buy one. You should then replicate \((-1)\) times the option. You should end up with a profit of \(0.6427 \times (1+r)^4 \approx 0.9410\) at time 4, or 0.6427 at time 0.

At time 1, since \(S_1 = 140\), you should hold the option, plus $1.336 and \(-0.0262\) shares. Notice that, despite having short-sold some stock, you owe money to the bank: the profit from short-selling stock wasn’t enough to buy the option.

Time 2: since \(S_2 = 70\), you want to hold \(-0.0866\) units of stock, so you short-sell an extra 0.0604 shares. You should now be holding 3.5325 in cash, \(-0.0866\) shares of stock, and 1 option.

If the stock price goes down again at time 3, you know the contract is worthless, so you buy back your stock and are left with the profit. If it goes up, you stick to the replicating portfolio.

4. As in the lecture, let \(S_0 = 80, \; r = 0, \; U = 1.5, \; D = 0.5\). In this exercise, you will derive the no-arbitrage price for the European put option with strike \(K = 80\) and expiration \(T = 3\). Recall that the buyer of the European put receives the option, but not the obligation, to sell one unit of the underlying stock at time \(T\) for \(K\). Its payoff at time \(T\) is

\[
P(K,T) = \begin{cases} 
K - S_T, & S_T < K; \\
0, & S_T \geq K.
\end{cases}
\]

(a) Derive the price of the European put at time 0 using risk-neutral valuation.

(b) Compute the replicating portfolios for all the possible outcomes at time \(T = 2\). Compare to the portfolios we computed in lecture for the European call. Do you notice anything interesting?

(c) Draw a graph to show

\[ P(K,T) = K - S_T + C(K,T). \]

Explain why this gives a way to price \(P(K,T)\) relative to \(C(K,T)\), and use this call-put parity to check your answer to part (4a).

Solution:

(a) The risk-neutral measure has \(P(U) = P(D) = 1/2\). The possibilities for the stock price at \(T = 3\) are 270 (contract worth 0), 90 (contract still worth 0), 30 (contract worth 50), or 10 (contract worth 70), so the value at time 0 is

\[ V_0 = (1-p)^3 \cdot 70 + 2p(1-p)^2 \cdot 50 = 27.5. \]
(b) Here are the replicating portfolios:

\[
\begin{align*}
\begin{cases}
  x + 270y = 0 & x = 0 \\
  x + 90y = 0 & x = 0 \\
  x + 90y = 0 & x = 75, y = -5/6; \\
  x + 30y = 50 & x = 60y = 25 \\
  x + 30y = 50 & x = 80, y = -1; \\
  x + 10y = 70 & x = 20y = 60
\end{cases}
\]

An important thing to notice is that whereas for the call, we usually had \( x < 0 \) and \( y > 0 \), we now have \( x > 0 \) and \( y < 0 \). A call is similar to a positive number of shares of stock: a long position in a call option is similar to a long position in the underlying stock. A put is similar to a negative number of shares: a long position in a put is similar to a short position in the underlying stock. Another way of thinking about the same thing: a call is worth more when the stock is more expensive, whereas a put is worth more when the stock is cheap. This holds for quite general calls and puts.

Actually the \( y \)-values here are always 1 less than the \( y \)-values in the lecture, and the \( x \)-values are 80 less than the \( x \)-values in the lecture. This lines up with the second part of the question: the sum of the call and the put should be the strike minus one share.

(c) What graph you draw is largely a matter of taste, but we have

\[
P(K,T) = \begin{cases}
  K - S_T, & S_T < K; \\
  0, & S_T > K
\end{cases}
\]

and

\[
K - S_T + C(K,T) = \begin{cases}
  K - S_T + 0, & S_T < K; \\
  K - S_T + S_T - K, & S_T > K
\end{cases}
\]

so the two are equal.

Here’s a plot that shows this:

The equality “put = call + $K - one share” should hold at all earlier times, so the value of a put at time \( t \) is given by

\[
V_t = V_t(\text{call}) + \frac{K}{(1 + r)^{T-t}} - S_t.
\]

Our values in part (a) and in the lecture are in agreement: \( 27.5 = 27.5 + 80/1^3 - 80 \). The call and put values match because the strike is equal to the initial stock price.