1. Let $S$ be the set plotted below, consisting of the union of the three ellipses. The $\sigma$-algebra $\mathcal{F}$ on $S$ contains the red ellipse (which is the union of regions $A, B, C, D$), the green ellipse (the union of $E, F, B, K, D$), and the blue ellipse (the union of $L, M, C, K, D$).

In this problem, you should ignore what happens to the boundaries of the regions.

(a) Using the fact that unions, intersections, and complements of sets in $\mathcal{F}$ must also be in $\mathcal{F}$, list all the sets that must be elements of $\mathcal{F}$. For example, the region $D$ is in $\mathcal{F}$, because it is the intersection of the three ellipses.

(b) Let the measure $m$ be given as follows: $m($red ellipse$) = 13$, $m($green ellipse$) = 15$, $m($blue ellipse$) = 20$, $m(S) = 40$, $m(B) = 3$, $m(K) = 2$, and $m(D) = 1$. Which other labelled regions are measurable? Using the additivity of measure, compute the measure of the regions that are measurable.

Solution:

(a) The smallest measurable sets are $A$, $B$, $C$, $D$, $E \cup F$, $K$, and $L \cup M$. For example, the set $A$ is the set of points that are red, but not blue and not green, so we can write

$$A = (\text{red}) \cap (\text{blue})^c \cap (\text{green})^c.$$  

This shows that $A$ belongs to the $\sigma$-algebra. To see that $E$ and $F$ individually don’t belong to the $\sigma$-algebra, notice that all the sets we have generating the $\sigma$-algebra contain either both $E$ and $F$ or neither of them, and this will be preserved by taking unions, intersections, and complements. Thus, $E \cup F$ is measurable, but neither $E$ nor $F$ individually are measurable.

Finally, all measurable sets are unions of 0 or more of the seven sets above; there are $2^7$ measurable sets in this $\sigma$-algebra. Note that we don’t have to include complements separately, because the complement of some of these regions is a union of the other regions.

(b) All of the sets in the above $\sigma$-algebra will be measurable; it suffices to compute the measure of the 7 basic regions.

Because $m($blue$) = m(E \cup F) + m(B) + m(K) + m(D)$, we can find $m(E \cup F) = 14$.

To compute $A$, $C$, and $L \cup M$, we’ll need to use $m(S)$. Because $S$ is the disjoint union of the seven basic regions, we can write

$$m(S) = m(A) + m(B) + m(C) + m(D) + m(E \cup F) + m(K) + m(L \cup M)$$

$$= (m(A) + m(B) + m(C) + m(D)) + (m(C) + m(D) + m(K) + m(L \cup M))$$

$$+ m(E \cup F) - m(C) - m(D).$$

Check that on the right-hand side we count the measure of each region exactly once!
This means that \( m(C) = 1 \), and we can find the rest of the measures:

\[
m(A) = 8, \quad m(B) = 3, \quad m(C) = 1, \quad m(D) = 1, \quad m(E \cup F) = 14, \quad m(K) = 2, \quad m(L \cup M) = 11.
\]

The measure of a set that’s a union of some of these intervals can be found by adding the measures of the intervals comprising it.

2. Let \( X : \mathbb{R} \to \mathbb{R} \) have the CDF given given below. What values does \( X \) take? What is the probability \( P(X = -1) \)?

![Graph of CDF](image)

**Solution:** Because the CDF is either flat or jumps, the only values of \( X \) are the jump points, so \( X = -1 \) or 1 or 2. To compute \( P(X = -1) \), we write

\[
P(X = -1) = P(X \leq -1) - P(X < -1) = F_X(-1) - \lim_{\epsilon \to 0} F_X(-1 - \epsilon) = \frac{1}{5}.
\]

The number can be read off the graph by noticing that 5 tick marks go into 1, so one tick mark must correspond to 1/5.

3. Match the following measures on \( \mathbb{R} \) to the plots of their CDFs.

(a) The uniform probability measure on the interval \([-1, 1]\).

(b) The uniform probability measure on the set \([-1, 1] \cup [3, 5]\).

(c) The measure \( m(A) = l(A \cap [-1, 1]) \), where \( l \) is the length or Lebesgue measure.

(d) The measure assigning mass 1 to the integers 1, 2, 3 and mass 0 to \( \{1, 2, 3\}^c \).

(e) The measure which is uniform measure of total mass 1 on \([-1, 1]\) and in addition assigns mass 3 to the set \{1\}. Formally:

\[
m(A) = \begin{cases} 
\frac{l([-1, 1] \cap A)}{l([-1, 1])}, & 1 \notin A \\
3 + \frac{l([-1, 1] \cap A)}{l([-1, 1])}, & 1 \in A
\end{cases}
\]

where \( l \) is the “length” or Lebesgue measure.
Solution: (a) maps to (D), (b) maps to (B), (c) maps to (A), (d) maps to (E), and (e) maps to (C).

Some things to watch out for:

(a) You should know the CDF of a uniform distribution on an interval. It’s 0 to the left of the leftmost endpoint (because none of the interval falls in \((-\infty, x]\)), 1 to the right of the rightmost endpoint (because all of the interval falls in \((-\infty, x]\)), and linear in between.

(b) Notice that \(X\) can’t be between 1 and 3, so the CDF is constant there. It’s 0 to the left of \(-1\), and 1 to the right of 5. (The fact that the final value is 1 comes from the fact that we’re talking about a probability measure.)

(c) Here we’re not talking about a probability measure, and so the final value is \(l([-1, 1]) = 2\).

(d) This is a counting measure: \(m((-\infty, x])\) counts how many of the numbers 1, 2, 3 are \(\leq x\). Hence the CDF has jump points when \(x = 1, 2, 3\).

(e) This is really not as scary as it looks! First, we can simplify \(l([-1, 1]) = 2\) in the denominator. Now let’s compute \(m((-\infty, x])\):

\[
m((-\infty, x]) = \begin{cases} 
\frac{1}{2}l(\emptyset) = 0, & x < -1; \\
\frac{1}{2}l([-1, x]) = \frac{1}{2}(x - (-1)) = \frac{1}{2}(x + 1), & -1 \leq x < 1; \\
3 + \frac{1}{2}l([-1, 1]) = 4, & x \geq 1 
\end{cases}
\]

4. A graph is a diagram with vertices and edges (an edge connects a pair of vertices). For example:

Let \(X\) be a map from the set of graphs to \(\mathbb{R}\), \(X(G) = \) number of vertices in \(G\), and let \(Y\) be the map counting the edges of \(G\). What are the values of \(X(G)\) and \(Y(G)\) if \(G\) is the graph shown in the diagram? Is \(X\) measurable with respect to \(\sigma(Y)\)?

Solution: The graph in the diagram has 6 vertices and 8 edges, so \(X(G) = 6\) and \(Y(G) = 8\). \(X\) is not \(\sigma(Y)\)-measurable: knowing how many edges the graph has doesn’t tell me how many vertices it
has. For example, here are two graphs with two edges:

\[ \circ \circ, \quad \circ - \circ - \circ \]

This means that \( X^{-1}(3) \) will not be a union of sets in \( \sigma(Y) \).

As several people pointed out, here’s one way to make \( X \) measurable with respect to \( \sigma(Y) \): require that we only look at complete graphs, i.e. graphs where every pair of vertices is connected by exactly one edge. Then \( X = n \) if and only if \( Y = \binom{n}{2} = \frac{1}{2} n(n - 1) \) (the binomial coefficient is read “\( n \) choose 2”). But for generic graphs, knowing \( Y \) doesn’t tell you much about \( X \) (nor does knowing \( X \) tell you much about \( Y \)).

5. Let \( Y : \{U, D\}^3 \to \mathbb{R} \) be defined as follows. For each \( i = 1, \ldots, 3 \), if the \( i \)th letter of the sequence is \( U \) let \( X_i = 3^{-i} \), else let \( X_i = 0 \); and let \( Y = \sum_i X_i \). For example:

\[ Y(UUD) = 3^{-1} + 3^{-2}. \]

What is \( \sigma(Y) \)?

**Solution:** Let’s compute \( Y(x) \) for every \( s \in \{U, D\}^3 \):

\[
\begin{align*}
Y(DDD) &= 0 & Y(DDU) &= 3^{-3} = \frac{1}{27} \\
Y(DUD) &= 3^{-2} = \frac{3}{9} = \frac{3}{27} & Y(DUU) &= 3^{-2} + 3^{-3} = \frac{4}{27} \\
Y(UDD) &= 3^{-1} = \frac{9}{3} = \frac{9}{27} & Y(UDU) &= 3^{-1} + 3^{-3} = \frac{10}{27} \\
Y(UUD) &= 3^{-1} + 3^{-2} = \frac{4}{9} = \frac{12}{27} & Y(UUU) &= 3^{-1} + 3^{-2} + 3^{-3} = \frac{13}{27}.
\end{align*}
\]

Because \( Y(s) \) is distinct for every \( s \in \{U, D\}^3 \), we see that each one-element set is in \( \sigma(Y) \). For example, \( \{UDD\} = Y^{-1}(\binom{3}{2}) \) is the preimage of a Borel set.

Because \( \{U, D\}^3 \) has only finitely many elements, and each one-element set is in \( \sigma(Y) \), we see that in fact all subsets of \( \{U, D\}^3 \) are in \( \sigma(Y) \) (all subsets are unions of finitely many one-element sets). Thus, \( \sigma(Y) = 2^S \) has \( 2^8 = 256 \) elements.

6. Match the random variables to the CDF of the measure they induce on \( \mathbb{R} \). (There may be many variables that induce the same measure on \( \mathbb{R} \), so there may be many variables with the same CDF.)

(a) \( S = \{U, D\}^2 \) with the uniform probability measure, \( X(s) \) is the index of \( s \) in alphabetical order (so \( X(DD) = 1 \) and \( X(UU) = 4 \)).

(b) \( S = \{U, D\}^2 \) with the uniform probability measure, \( X(s) \) is the number of \( U \)'s in the sequence.

(c) \( S = \{\text{red}, \text{green}, \text{blue}\} \). (This is a set of three elements, not the Venn diagram of question 1.) The measure is defined by \( m(\{\text{red}\}) = \frac{1}{3}, m(\{\text{green}\}) = m(\{\text{blue}\}) = 1/4 \). The random variable is defined by \( X(\text{green}) = 0, X(\text{red}) = 1, X(\text{blue}) = 2 \).

(d) \( S = \{U, D\}^2 \) with the probability measure given by \( m(\{UU, DD\}) = \frac{1}{2} \) and \( m(\{UD, DU\}) = \frac{1}{2} \). The random variable is \( X(DD) = X(UU) = 0, X(UD) = X(DU) = 1 \).

(e) \( S = \{U, D\}^2 \) with the same measure as above, but the random variable is \( X(s) = 1 \) for all \( s \in S \).

(f) \( S \) is the Venn diagram from question 1, and the measure is given by \( m/40 \) where \( m \) is the measure from question 1. \( X \) counts the number of ellipses that a point belongs to. For example, \( X(s) = 1 \) if \( s \in L \), and \( X(s) = 3 \) if \( s \in D \).

(g) \( S = [0, 5] \) with the uniform probability measure, and \( X(s) = s^2 \).
Solution:

(a) The answer is (E). There are 4 elements of $S$, they are each equally likely, and $X$ maps them to distinct numbers 1 through 4. This means that the CDF of $X$ will have jumps at 1, 2, 3, 4 of equal size $1/4$.

(b) The answer is (B). There are 4 elements of $S$, they are equally likely, but $X$ maps one of them to 0, two of them to 1, and one of them to 2. Hence, for example, $P(X \leq 1) = P(\{DD, DU, UD\}) = \frac{3}{4}$. The CDF of $X$ will have jumps at 0, 1, and 2, but the jump at 1 will be large (size $1/2$, corresponding to two extra elements of $S$) and the jumps at 0 and 2 will be small (size $1/4$).

(c) The answer is again (B). $X$ takes on the values 0, 1, 2 with probabilities $P(\text{green}) = 1/4$, $P(\text{red}) = 1/2$, and $P(\text{blue}) = 1/4$ respectively, just like in the previous part.

(d) The answer is (A). $X$ takes on the values 0 and 1, with probability $1/2$ each, so the CDF of $X$ has two jumps (one at 0 and the other at 1), each of size $1/2$.

(e) The answer is (D). If $X$ is always 1, then $P(X \leq x) = 0$ if $x < 1$, and $P(X \leq x) = 1$ of $x \geq 1$.

(f) The answer is (F). $X$ takes on the values 1 through 3, and for example,
\[ P(X = 1) = \frac{1}{40} (m(A) + m(E \cup F) + m(K \cup L)) \approx 0.8. \]

Thus, the CDF should have jumps at 1, 2, and 3, and at 1 it should jump by about 0.8.

(g) The answer is (C). Let’s compute:
\[ F_X(x) = P(X \leq x) = P(\{s : s^2 \leq x\}) = P(\{s \leq \sqrt{x}\}). \]

This will be 0 if $x < 0$, $\frac{1}{5} \sqrt{x}$ if $x \in [0, 25]$, and 1 if $x \geq 25$. Make sure you’re OK with plotting the graph of $\sqrt{x}$!

7. Below are three plots of random variables $X, Y : \mathbb{R} \to \mathbb{R}$. The graph of $X$ is in black; the graph of $Y$ is in red. (I think the difference should show up even on a black-and-white printout.) For each plot, decide whether $Y$ is $\sigma(X)$-measurable, $X$ is $\sigma(Y)$-measurable, both, or neither. Ignore what happens for values outside the plot.
Solution:

(a) $X$ is $\sigma(Y)$-measurable: if I tell you how high the red graph is (e.g. I tell you $Y = 0$), then you can figure out what $X$ must be. On the other hand, $Y$ is not $\sigma(X)$-measurable, because if I tell you that $X = 1$, you could have $Y = 0$ or $Y = -1$. That is, matched to a single value of $X$ there are multiple values of $Y$.

(b) This depends on the details of the graph, but the intention was for neither random variable to be measurable with respect to the other. It’s clear that $X$ is not $\sigma(Y)$-measurable: there are all sorts of $X$-values associated to the constant part of the $Y$ graph. To see that $Y$ is also not $\sigma(X)$-measurable, look at the right-hand end of the middle section of the $X$ plot: the $Y$ value above that is close to 0. On the other hand, if you look at the left-hand end of the $X$ plot, you can find the same $X$ value, with a much larger $Y$ value. See the picture below.

(c) Here, $Y$ is $\sigma(X)$-measurable, but $X$ is not $\sigma(Y)$-measurable. Knowing which segment I’m on for the black plot lets me figure out what segment I’m on for the red plot, but not conversely.

8. For each pair of random variables $(X, Y)$ below, decide whether $Y$ is $\sigma(X)$-measurable, $X$ is $\sigma(Y)$-measurable, both, or neither. Sketch the CDF of the law of $X$, making sure that the jump points (if any) are in the correct place. (You don’t need to be very precise.)

(a) $S = \{U, D\}^4$ with the uniform measure (i.e. each sequence has measure 1/16). $X : \{U, D\}^4 \to \{1, \ldots, 16\}$ is the index of the sequence in alphabetical order, $Y : \{U, D\}^4 \to \mathbb{R}$ is 1 when the second letter is a $D$, and 0 otherwise.

(b) $S = \mathbb{Z}_+$ with the probability measure $P(n) = e^{-\lambda} n^{-1} / n!$, $n \geq 0$. $X : S \to \mathbb{R}$ is given by $X(n) = n^2$, $Y : S \to \mathbb{R}$ is given by $Y(n) = n$.

(c) $S = \mathbb{Z}$ with the probability measure given by $P(0) = 0$, $P(\pm n) = \frac{1}{2} e^{-\lambda} |n|^{-1} / |n|!$. $X : S \to \mathbb{R}$ is given by $X(n) = n^2$, $Y : S \to \mathbb{R}$ is given by $Y(n) = n$.

(d) $S = [-1, 1] \times [-1, 1]$ with the uniform measure, $X((x, y)) = x$, $Y((x, y)) = y$.

(e) $S$ = unit circle in $\mathbb{R}^2$ with the uniform measure, $X((x, y)) = x$, $Y((x, y)) = y$. (Note: the unit circle is the set of points $(x, y)$ with $x^2 + y^2 = 1$.)

(f) $S$ = unit disk in $\mathbb{R}^2$ with the uniform measure, $X((x, y)) = x$, $Y((x, y)) = y$. (Note: the unit disk is the set of points $(x, y)$ with $x^2 + y^2 \leq 1$.)

Solution: All plots are at the end of the solution.

(a) Knowing $X$ uniquely identifies the sequence of four letters, and therefore lets me compute $Y$. On the other hand, knowing $Y$ does not let me compute $X$, for example: $Y(DDDD) = Y(DDDU)$ but $X(DDDD) \neq X(DDDU)$. Thus, $Y$ is $\sigma(X)$-measurable, but $X$ is not $\sigma(Y)$-measurable.

The CDF of $X$ will be 0 for $x < 1$, 1 for $x > 16$, and will have jumps of size 1/16 at each of 1, 2, $\ldots$, 16. Between those numbers it will be flat.
(b) For measurability, we don’t are about \( P \). Clearly \( X \) is \( Y \)-measurable: if I know \( n \) then I know \( n^2 \). Here also \( Y \) is \( X \)-measurable, because \( n^2 \) has a unique non-negative square root. The CDF of \( X \) will be 0 for \( x < 0 \), it will never hit 1, and it will have jumps at each of \( 0^2, 1^2, 2^2, 3^2, \ldots \). (Note no jump at 2, since 2 is not a square.) It will be flat between integers. The size of the jumps depends on \( \lambda \), so it’s hard to even say which of the jumps should be larger than others, although eventually they’ll have to get pretty small.

(c) Here, \( X \) is still \( \sigma(Y) \)-measurable, but \( Y \) is not \( \sigma(X) \)-measurable: \( n^2 \) has two square roots \( n \) and \( -n \).

The probability measure on \( S \) was constructed from the distribution in the previous question: I placed mass 0 at 0, and half of the mass that used to be at \( n \) is now at each of \( n + 1 \), \(-n + 1 \). Since \( X(n + 1) = X(-n + 1) = (n + 1)^2 \), we’ll get the same levels as in the CDF in the previous question, but they’ll be shifted over to the next interval.

(d) Neither of \( X \) and \( Y \) is measurable with respect to the other: knowing the \( x \)-coordinate of a point in the square tells you nothing about its \( y \)-coordinate, and conversely. To compute the CDF, we have

\[
P(X \leq a) = \mathbb{P}(\{(x, y) \in [-1, 1]^2 : x \leq a\})
= \frac{1}{4} \cdot \text{area of rectangle } [-1, a] \times [-1, 1] = \frac{1}{2} (a + 1)
\]

assuming \(-1 \leq a \leq 1 \), and \( P(X \leq a) = 0 \) if \( a < -1 \), \( P(X \leq a = 1) \) if \( a < 1 \).

(e) Again neither of \( X \) and \( Y \) is measurable with respect to the other, because knowing the value of one only lets you determine the other up to sign. (But if we had said that \( X \) and \( Y \) must be nonnegative, they’d be measurable with respect to each other.) The CDF here is tricky! For \( a < -1 \) and \( a > 1 \) we of course have \( F_X(a) = 0 \) and \( F_X(a) = 1 \) respectively, but in between:

\[
P(X \leq a) = \frac{1}{2\pi} \cdot \text{length of arc of circle to the left of } x = a
= \frac{1}{2\pi} \cdot \text{length of arc of circle with angles from } \cos^{-1}(|a|) \text{ to } 2\pi - \cos^{-1}(|a|)
= \frac{1}{2\pi} \left( 2\pi - 2 \cos^{-1}(|a|) \right).
\]

We divide by \( 2\pi \) because that’s the circumference of the entire circle. See this diagram:

(f) And still neither of \( X \) and \( Y \) is measurable with respect to the other. For \( a < -1 \) and \( a > 1 \) we have the usual \( F_X(a) = 0 \) and \( F_X(a) = 1 \) respectively. In between, for \( a \leq 0 \):

\[
P(X \leq a) = \frac{1}{\pi} \cdot \text{area of portion of circle to the left of } x = a
= \frac{1}{\pi} \cdot \left( \text{area of wedge} - \text{area of triangle} \right)
= \frac{1}{\pi} \left( \frac{2\pi - 2 \cos^{-1}(|a|)}{2} - |a| \sqrt{1 - a^2} \right).
\]

We divide by \( \pi \) because that’s the area of the entire circle. See this diagram:
For $0 \leq a \leq 1$ you can skip the computations, and just notice that

$$\mathbb{P}(X > a) = \mathbb{P}(X < -a) \implies 1 - F_X(a) = F_X(-a).$$

Here are all the CDFs (left to right, then top to bottom). The last two are plotted using Maple, a computer algebra program; you could use Wolfram Alpha (free and online) instead, or MATLAB, or whatever you have access to.