1. Let $S = [0, 1]$ equipped with the uniform probability measure (and the Borel $\sigma$-algebra). Let $X_1$ and $X_2$ be defined on $S$ as follows:

$$X_1(s) = \begin{cases} 
\text{duck}, & 0 \leq s < \frac{1}{3} \\
\text{goose}, & \frac{1}{3} \leq s < \frac{2}{3} \\
\text{sheep}, & \frac{2}{3} \leq s \leq 1 
\end{cases}$$

$$X_2(s) = \begin{cases} 
2, & 0 \leq s < \frac{1}{4} \\
3, & \frac{1}{4} \leq s < \frac{1}{2} \\
2, & \frac{1}{2} \leq s < \frac{3}{4} \\
3, & \frac{3}{4} \leq s \leq 1.
\end{cases}$$

Are $X_1$ and $X_2$ independent?

Let $X_3$ be defined on $S$ as follows:

$$X_3(s) = \begin{cases} 
1, & 0 \leq s < \frac{1}{6} \\
2, & \frac{1}{6} \leq s < \frac{1}{3} \\
3, & \frac{1}{3} \leq s < \frac{1}{2} \\
4, & \frac{1}{2} \leq s < \frac{2}{3} \\
5, & \frac{2}{3} \leq s < \frac{5}{6} \\
6, & \frac{5}{6} \leq s \leq 1.
\end{cases}$$

Are $X_1$ and $X_3$ independent? Are $X_2$ and $X_3$?

Finally, let $X_4$ be defined on $S$ as follows:

$$X_4(s) = \begin{cases} 
1, & 0 \leq s < \frac{1}{6} \\
2, & \frac{1}{6} \leq s < \frac{1}{3} \\
1, & \frac{1}{3} \leq s < \frac{1}{2} \\
2, & \frac{1}{2} \leq s < \frac{2}{3} \\
1, & \frac{2}{3} \leq s < \frac{5}{6} \\
2, & \frac{5}{6} \leq s \leq 1.
\end{cases}$$

Check that $X_1$ and $X_4$ are independent.

Let $g : \{\text{duck, goose, sheep}\} \to \mathbb{R}$ be given by $g(\text{duck}) = 10$, $g(\text{goose}) = 15$, $g(\text{sheep}) = 50$. Compute $E[g(X_1) \cdot X_4]$ by working out the distribution of the real-valued random variable $g(X_1) \cdot X_4$, and check that $E[g(X_1)X_4] = E[g(X_1)]E[X_4]$.

2. For each pair of random variables $X, Y$ below, decide:

- Is $X$ $\sigma(Y)$-measurable? Is $Y$ $\sigma(X)$-measurable? Both? Neither?
- If $\mu$ is the law of $X$, and $\nu$ is the law of $Y$, is $\mu \ll \nu$? $\nu \ll \mu$? Both? Neither?
- Are $X$ and $Y$ independent?
- Are $X$ and $Y$ uncorrelated, that is, $\text{Cov}(X, Y) = 0$?

(Note: some of these questions may not be meaningful for a given pair of random variables.)

(a) Let $S = [0, 1]$ with the uniform probability measure, and let $X : S \to \mathbb{R}$ be given by

$$X(s) = [3s] = \begin{cases} 
0, & s < 1/3 \\
1, & 1/3 \leq s < 2/3 \\
2, & 2/3 \leq s < 1 \\
3, & s = 1.
\end{cases}$$
Let $Y : S \rightarrow \mathbb{R}$ be given by $Y(s) = s^2$.

(b) Let $S = \{U, D\}$ with the uniform probability measure, and let $X : S \rightarrow \mathbb{R}$ and $Y : S \rightarrow \mathbb{R}$ be given by

$$X(s) = \text{number of } U \text{'s in } s, \quad Y(s) = \text{number of } U \text{'s among the first three letters of } s.$$  

(Note: figuring out the expectation of $XY$ can be tricky!)

(c) Let $S = \mathbb{R}$, and let the probability measure on it be given by

$$m([a, b]) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$  

Let $X : S \rightarrow \mathbb{R}$ be given by $X(s) = s$, and let $Y : S \rightarrow \mathbb{R}$ be given by $Y(s) = s^2$.

(d) $S = [-1, 1] \times [-1, 1]$ with the uniform probability measure, $X : S \rightarrow \mathbb{R}$ is $X((s_1, s_2)) = s_1$, $Y : S \rightarrow \mathbb{R}$ is $Y((s_1, s_2)) = s_2$.  

(e) $S$ is the filled unit disk $\{(s_1, s_2) : s_1^2 + s_2^2 \leq 1\}$, with the uniform probability measure. $X : S \rightarrow \mathbb{R}$ is $X((s_1, s_2)) = s_1$, $Y : S \rightarrow \mathbb{R}$ is $Y((s_1, s_2)) = s_2$.  

(f) $S = \{1, \ldots, 6\}$ with the uniform probability measure, $X : S \rightarrow \mathbb{R}$ is given by $X(s) = (s - 2)^2$, $Y : S \rightarrow \{\text{cup, saucer, teapot}\}$ is given by

$$Y(s) = \begin{cases} 
\text{cup}, & s = 1 \text{ or } 3 \\
\text{saucer}, & s = 2 \\
\text{teapot}, & s \geq 4.
\end{cases}$$  

(Hint: you may find this easier if you write out the values of $X(s)$ first.)