1. Let $X : [0, 1]^2 \rightarrow \mathbb{R}$ be the random variable given by $X(s_1, s_2) = s_1 + s_2$. (The smallest $X$ could be is 0, and the largest $X$ could be is 2, but it’s not uniform on $[0, 2]$!) The space $[0, 1]^2$ is equipped with the uniform probability measure.

(a) Compute and plot the CDF of $X$. (Hint: draw the set \{$(s_1, s_2) : X(s_1, s_2) \leq a$\}. What’s its area?)

(b) Use your computation to find the density of $X$, if it has one.

(c) Describe $\sigma(X)$ in words, i.e. describe the information you obtain about the point $(s_1, s_2)$ by observing $X(s_1, s_2)$.

(d) Let $Y : [0, 1]^2 \rightarrow \mathbb{R}$ be given by $Y(s_1, s_2) = s_1$. Are $X$ and $Y$ independent?

(e) Describe $\sigma(Y)$, i.e. explain what information about $(s_1, s_2)$ you gain from observing $Y(s_1, s_2)$.

(f) Let $Z : [0, 1]^2 \rightarrow \mathbb{R}$ be given by $Z(s_1, s_2) = s_2$. Is $Z$ measurable with respect to $X$? With respect to $Y$? With respect to the two-dimensional random variable $(X, Y)$?

Solution:

(a) Here’s the plot of the set $X \leq a$:

\[ a + s_2 \leq a, \quad a > 1 \]
\[ a - 1 \leq s_1 \leq a, \quad a < 1 \]
\[ s_1 + s_2 \leq a, \quad a < 1 \]

It’s easy to see that the area is $\frac{1}{2}a^2$ if $0 \leq a \leq 1$, and $1 - \frac{1}{2}(1-a)^2$ if $1 \leq a \leq 2$. Thus,

\[ \mathbb{P}(X \leq a) = \begin{cases} 
0, & a \leq 0 \\
\frac{1}{2}a^2, & 0 \leq a \leq 1 \\
1 - \frac{1}{2}(1-a)^2, & 1 \leq a \leq 2 \\
1, & a \geq 2
\end{cases} \]

(Notice that the two expressions agree at 0, 1, and 2.) The plot is sketched below:

(b) The density is the derivative of the CDF: $f_X(a) = a$ for $0 \leq a \leq 1$, then $1 - a$. (The plot looks triangular)

(c) $\sigma(X)$ consists of diagonal strips, aligned from northwest to southeast. The value $X(s_1, s_2)$ identifies a diagonal on which the point $(s_1, s_2)$ lies.

(d) $X$ and $Y$ are not independent. One way to see this is to notice that $X \leq 1 + Y$, so $\mathbb{P}(Y < 0.3, X > 1.5) = 0$ while $\mathbb{P}(Y < 0.3)$ and $\mathbb{P}(X > 1.5)$ are both positive.

(e) $\sigma(Y)$ is generated by vertical strips, because $Y$ tells you about the first coordinate but not the second one.
Z is not X-measurable, since Z is about the second coordinate, and each value of X corresponds to multiple values of Z. Z is not Y-measurable either, in fact it’s independent of Y. However, Z is (X,Y)-measurable, because Z = X − Y (geometrically, because if I give you a diagonal and a vertical line, you can figure out their point of intersection, and get its second coordinate).

2. Let (X, Y) be jointly normal, with mean \((0, 0)\) and covariance matrix \(\Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\). Write down the joint density of X and Y.

Let Z be a standard normal random variable independent of X. Find a linear combination of X and Z (i.e. \(aX + bZ\)) that has the same mean and variance as Y, and the same covariance with X.

This lets you represent \((X, Y)\) as \((X, aX + bZ)\) where Z is independent of X. The decomposition into a measurable and an independent part is often useful.

**Solution:** Let Z be a standard normal random variable that’s independent of X. Clearly, any linear combination of X and Z has mean 0, just like Y. It remains to solve 

\[
\text{Var}[aX + bZ] = \text{Var}Y, \quad \text{Cov}[aX + bZ, X] = \text{Cov}(Y, X).
\]

Once we do that, we’ll see that \((X, aX + bZ)\) are jointly normal (as a linear transformation of jointly normal random variables \((X, Z)\)), and hence have the same distribution as \((X, Y)\) (because the distribution of jointly normal random variables depends only on the means and the covariance matrix).

Solving, we have 

\[
a^2 + b^2 = 3, \quad a = 1
\]

(recall that X and Z are independent). So, the linear combination we want is \(X + \sqrt{2}Z\).

3. Let \((X_1, X_2, X_3)\) be jointly normal with mean and covariance

\[
\mu = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}.
\]

(a) Are \(X_1\) and \(X_2\) independent? What is their joint distribution?

(b) Are \(X_1 + X_2\) and \(X_1 - X_2\) independent? What is their joint distribution?

(c) Is \(X_3\) independent of the two-dimensional random variable \((X_1, X_2)\)?

**Solution:**

(a) Yes. They are jointly normal, \(X_1\) has mean 0 and variance 3, \(X_2\) has mean 1 and variance 1, and since they’re jointly normal and have covariance 0, they’re independent. The joint density is 

\[
f_{X_1, X_2}(a, b) = \frac{1}{\sqrt{2\pi}\sqrt{3}}e^{-\frac{1}{2}a^2/3} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(b-1)^2}.
\]

(b) No. They are jointly normal with mean \((1, -1)^T\), variances 4 (for each of them), and covariance 

\[
\text{Cov}(X_1 + X_2, X_1 - X_2) = \text{Var}(X_1) - \text{Var}(X_2) = 2.
\]

Since covariance is nonzero, they’re not independent. The joint density is 

\[
f_{X_1, X_2}(a, b) = \frac{1}{\sqrt{2\pi} \cdot 12} \exp\left(-\frac{1}{2} \left(\frac{1}{6}(a - 1)^2 + \frac{1}{6}(b + 1)^2 - \frac{2}{3}(a - 1)(b + 1)\right)\right).
\]

I’m using the matrix inverse 

\[
\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}.
\]
(c) No, $X_3$ isn’t even independent of $X_1$ (since they have a correlation), so it can’t be independent of $(X_1, X_2)$.

4. Let $(B_t, t \geq 0)$ be a Brownian motion. Find:
   (a) $E[B_{17}^3]$
   (b) $\text{Cov}(B_3, B_7)$
   (c) the distribution of $B_{10} + B_{20} - B_{30}$.

   **Solution:**
   (a) $B_{17}$ is a mean-zero normal random variable, its third moment is zero by symmetry.
   (b) $\text{Cov}(B_3, B_7) = 3$.
   (c) $B_{10} + B_{20} - B_{30}$ is normal (because it’s a linear combination of jointly normal random variables), with mean 0, and variance
   \[
   \text{Cov}(B_{10} + B_{20} - B_{30}, B_{10} + B_{20} - B_{30}) = 10 + 20 + 30 + 2 \cdot \text{Cov}(B_{10}, B_{20}) - 2 \cdot \text{Cov}(B_{10}, B_{30}) - 2 \cdot \text{Cov}(B_{20}, B_{30}) = 10 + 20 + 30 + 2 \cdot 10 - 2 \cdot 20 = 20.
   \]

5. Let $B_t$ be a Brownian motion. Find
   (a) $E[B_t^2 B_s - B_t^3 | (t \geq s)]$ (assume $t \geq s$).
   (b) $E[(B_4 - B_3)^2 + (B_3 - B_2)^2 + (B_2 - B_1)^2 + (B_1 - B_0)^2]$.

   **Solution:**
   (a) $E[B_t^2 B_s - B_t^3 | (t \geq s)] = E[B_s^2 | B_t] = 0 + E[B_s^2 | B_t] = 0$. (The last equality is because Brownian motion has independent increments.) $E[B_t^3] = 0$ also. Thus, the answer is zero.
   (b) Each term has expected value 1 (that’s the variance of the increment of the Brownian motion), so the total expectation is 4.

6. Let $Z \sim N(0, 1)$ and let $X_t = \sqrt{t} Z$. What is the distribution of $X_t$? Is this process a Brownian motion?

   **Solution:** For each $t$, $X_t \sim N(0, t)$. However, the increments of this process are not independent: for example, $X_3 - X_2 = (\sqrt{3} - \sqrt{2})Z$ is not independent of $X_2$ (in fact, it’s measurable with respect to it). So no, this is not a Brownian motion.

7. A **Brownian bridge** is a Gaussian stochastic process $(X_t, 0 \leq t \leq 1)$ defined as follows: $E[X_t] = 0$, and $\text{Cov}(X_t, X_s) = \min(t, s) - t u$. (Recall that this is enough to define a Gaussian process.) Show that if $(B_t, t \geq 0)$ is a Brownian motion, then $Y_t = B_t - t B_1$ is a Brownian bridge. (Recall: it’s enough to check that $Y_t$ is a Gaussian process with the correct mean and variance.)

   **Solution:** We need to check that $(Y_{t_1}, \ldots, Y_{t_n})$ are jointly normal; and then we need to check that the mean and covariance structure are right.
   Joint normality:
   \[
   (Y_{t_1}, \ldots, Y_{t_n}) = (B_{t_1} - t_1 B_1, \ldots, B_{t_n} - t_n B_1)
   \]
   is a linear transformation of the jointly normal vector $(B_{t_1}, \ldots, B_{t_n}, B_1)$, hence jointly normal.
   Mean: $E[Y_t] = E[B_t] - tE[B_1] = 0 - 0 = 0$. 

3
Covariance: suppose $t \leq u$, then

$$\text{Cov}(Y_t, Y_u) = \text{Cov}(B_t - tB_1, B_u - uB_1)$$

$$= \text{Cov}(B_t, B_u) - t \text{Cov}(B_u, B_1) - u \text{Cov}(B_1, B_1) + tu \text{Var}(B_1)$$

$$= t - tu - ut + tu = t - tu.$$