1. (20 pts) Consider the binomial market model with one stock (whose prices are denoted \( S_t \)). Let the interest rate be \( r = 0.5 \). (This isn’t a very realistic model!) Let \( S_0 = $100 \), and suppose

\[
S_{t+1} = \begin{cases} 
1.7S_t, & \text{or} \\
1.2S_t, & \text{or} 
\end{cases}
\]

Consider a European call option \( C(189, 2) \) with strike price \( K = $189 \) and execution time \( T = 2 \).

(a) Compute the value of the contract at time 0; show your work.

(b) Find the replicating portfolio at time 0. (If you don’t have a calculator, you can write down a purely numeric expression like \((123 - 45 \cdot 6)/78.9\), without simplifying it down to a single number.)

**Solution:**

(a) The risk-neutral measure is up 3/5, down 2/5. Therefore, the value is

\[
\frac{(0.6^2 \cdot (289 - 189) + 2 \cdot 0.6 \cdot 0.4 \cdot (204 - 189))}{1.5^2} = 19.2
\]

(b) The two possible values at time 1 are

\[
V_{1|S_1=170} = \frac{0.6 \cdot 100 + 0.4 \cdot 15}{1.5} = 44 \quad \text{and} \quad V_{1|S_1=120} = \frac{0.6 \cdot 15 + 0.4 \cdot 0}{1.5} = 6.
\]

The replicating portfolio consists of \( x \) bonds and \( y \) shares of stock, where \( x \) and \( y \) satisfy

\[
\begin{cases} 
1.5x + 170y = 44 \\
1.5x + 120y = 6
\end{cases}
\]

from which \( 50y = 38 \) so \( y = 0.76 \), and

\[
x = \frac{6 - 120 \cdot 0.76}{1.5} = -56.8.
\]

(Another way to find \( x \) is to notice that \( x + 0.76S_0 = V_0 = 19.2 \).)

2. (15 pts) Let \( \mu \), \( \nu \), and \( \eta \) be measures on \( S = \{U, D\}^2 \) defined as follows.\(^1\) Let \( \mu \) be the uniform probability measure. Let \( \nu \) be the risk-neutral measure from the previous problem. Let \( \eta(\{s\}) = 1/4 \times \text{number of } U \text{'s in } s \). For each of the three pairs (\( \mu \) and \( \nu \), \( \mu \) and \( \eta \), \( \nu \) and \( \eta \)) determine whether they are absolutely continuous with respect to each other. (There are six absolute continuities to check, since you need to check each pair of measures in each direction.) Justify your answer briefly.

**Solution:** Since both \( \mu \) and \( \nu \) assign positive probabilities to all the one-point sets, we have \( \mu \sim \nu \) and \( \eta \ll \mu, \eta \ll \nu \). The last two are not reversible, because \( \eta(\{DD\}) = 0 \) while \( \mu(\{DD\}) > 0, \nu(\{DD\}) > 0 \).

3. (13 pts) Let \( X \sim N(2, 4) \) be a normal random variable with mean 2 and variance 4 (standard deviation 2). Let \( Y = \exp(2 + 3X) \). Compute \( \text{Var}[Y] \).

You may find the following formula useful (for any \( \mu \in \mathbb{R} \) and \( \sigma^2 > 0 \)):

\[
\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} \, dx = e^{\mu + \frac{1}{2} \sigma^2}.
\]

\(^1\mu \) is pronounced “mu”, \( \nu \) is pronounced “nu”, and \( \eta \) is pronounced “eta.”
Solution: Note that the formula gives you the value of $\mathbb{E}[e^Y(\mu, \sigma^2)]$.
Now,
\[
\text{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = \mathbb{E}[e^{4+6X}] - (\mathbb{E}[e^{2+3X}])^2
= \mathbb{E}[e^{N(16, 144)}] - (\mathbb{E}[e^{N(8, 36)}])^2
= e^{16+\frac{3}{2}144} - (e^{8+\frac{1}{2}36})^2 = e^{88} - e^{52}.
\]
I’m freely using the fact that linear transformations of $X$ are still normal, with the appropriate mean and variance.

4. (30 pts) For each pair of random variables $(X, Y)$ below, decide four things:
   - Is $Y$ is $\sigma(X)$-measurable, is $X$ is $\sigma(Y)$-measurable, both, or neither?
   - Are $X$ and $Y$ independent?
   - Are $X$ and $Y$ uncorrelated? (Hint: you don’t always have to compute the value of the covariance in order to know that it’s non-zero!)
   - Let $\mu$ be the distribution of $X$, and let $\nu$ be the distribution of $Y$. Is $\mu \ll \nu$, $\nu \ll \mu$, both (i.e. $\mu \sim \nu$), or neither?

Note that some of these questions may not make sense for the variable pair; if that is the case, you should say so. Justify your answers briefly.

(a) Let $S = \{U, D\}^4$ with the uniform probability measure. $X : S \to \{1, \ldots, 16\}$ is the index of the sequence in alphabetical order, $Y : \{U, D\}^4 \to \mathbb{R}$ is 1 when the second letter is a $D$ and 0 otherwise.
   Solution: $Y$ is $\sigma(X)$-measurable, $X$ is not $\sigma(Y)$-measurable. Not independent, since $Y$ is $\sigma(X)$-measurable and non-constant. Not uncorrelated: we have $Y = 1$ if $X = 1, \ldots, 4$ or $X = 9, \ldots, 12$, and $Y = 0$ if $X = 5, \ldots, 8$ and $13, \ldots, 16$, so $\mathbb{E}[XY] = \frac{1}{2}(2.5+10.5) = 6.5$, whereas $\mathbb{E}[X]\mathbb{E}[Y] = 8.5\frac{1}{2} = 4.25$. The negative correlation tells you that $X$ tends to be smaller when $Y = 1$ than when $Y = 0$. Not absolutely continuous in either direction: $F_\mu$ jumps at 1 through 16 and $F_\nu$ jumps at 0 and 1.

(b) Let $S = [0, 1]$ and $T = [0, 10]$, each with the uniform probability measure. Let $X : S \to \mathbb{R}$ be given by $X(s) = 10s^2 - 3$ and let $Y : T \to \mathbb{R}$ be given by $Y(t) = 7 - t$.
   Solution: Measurability, independence, correlation don’t make sense since $X$ and $Y$ have no joint behavior. The measures are equivalent, since $X$ and $Y$ both take values on the interval $[-3, 7]$ and have densities there.

(c) Let $S = \{1, 2, 3, 4\}$, with the uniform probability measure. Let
   $X(1) = \text{“one”, } X(2) = \text{“two”, } X(3) = \text{“three”, } X(4) = \text{“four”}$.
   Let $Y(s) = 4 - s$.
   Solution: Measurable in both directions, since values map one-to-one. Not independent, since measurable and non-constant. Correlation doesn’t make sense since $X$ is not numeric. Absolute continuity doesn’t make sense since $X$ and $Y$ take values in different sets.

5. (12 pts) Determine whether the following statements are true or false. You do not need to justify your answer.
   (a) If $\mathcal{F}$ is a $\sigma$-algebra, and both $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then always $A \cap B \in \mathcal{F}$. T
   (b) If $S$ has 5 elements, then every $\sigma$-algebra on $S$ has at least 2 and at most $2^5 = 32$ elements. T
   (c) If $f$ is the density of a real-valued random variable $X$, then $f$ is nondecreasing. F, this is true for CDF.
   (d) If $X$, $Y$, and $Z$ are independent, then $E[X^3Y - Z] = E[X^3]E[Y] - E[Z]$. T
(e) If \(X\) is \(\sigma(Y)\)-measurable, then the law of \(X\) must be absolutely continuous with respect to the law of \(Y\).

(f) If \(X\) has density \(f_X\) on the real line and \(Y\) has density \(f_Y\), then

\[
\mathbb{E}[\sin(X)e^{3Y+7}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin(a)e^{3b+7} f_X(a)f_Y(b) \, da \, db.
\]

F, this is true if \(X\) and \(Y\) are independent.