1. (15 pts) In the Black-Scholes model, stock price is given by $S_t = e^{\sigma B_t + \mu t}$, and the interest rate is $r$. Below, $\tilde{E}$ and $\tilde{P}$ refer to the expectation and probability under the risk-neutral measure. Write down, but do not try to evaluate integral expressions that would compute the required expectations.

(a) Derive an expression for $S_t$ under the risk-neutral measure in terms of $\tilde{B}_t$, a Brownian motion. Show that your measure is equivalent and that under it, $e^{-rt}S_t$ is indeed a martingale.

(b) $\tilde{P}(S_t > K)$.

(c) $\tilde{E}[\sin S_t]$.

Solution:

(a) The original SDE that $e^{-rt}S_t$ satisfied is

$$d(e^{-rt}S_t) = \sigma S_t dB_t + (\mu - r + \frac{1}{2} \sigma^2) S_t dt.$$  

Changing measure so that $d\tilde{B}_t = dB_t + \frac{mu - r + \frac{1}{2} \sigma^2}{\sigma} dt$, $\tilde{B}_t = B_t + \frac{mu - r + \frac{1}{2} \sigma^2}{\sigma} t$ is a Brownian motion gives

$$d(e^{-rt}S_t) = \sigma t \tilde{B}_t,$$

showing that this is indeed a martingale.

Rewriting $S_t$ in terms of $\tilde{B}_t$ gives

$$S_t = e^{(r-\frac{1}{2} \sigma^2)t + \sigma \tilde{B}_t}.$$  

You can figure this out by matching coefficients in the differential equation and the exponential formulation.

(b)

$$\tilde{P}(S_t > K) = \tilde{P}(\log S_t > \log K) = \int_{\log K}^{\infty} \frac{1}{\sqrt{2\pi \sigma^2 t}} \exp\left(-\frac{1}{2 \sigma^2 t} (u - (r - \frac{1}{2} \sigma^2)t)^2\right) du.$$  

Here, I’m integrating over $(r - \frac{1}{2} \sigma^2)t + \sigma \tilde{B}_t \sim N((r - \frac{1}{2} \sigma^2)t, \sigma^2 t)$. Of course, you could integrate over $\tilde{B}_t$ instead, you’d just have to figure out what the range of integration is.

(c) Similarly,

$$\tilde{E}[\sin S_t] = \int_{-\infty}^{\infty} \sin \left(e^{(r-\frac{1}{2} \sigma^2)t + \sigma u}\right) \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}u^2} du.$$  

Here, I’m integrating over $\tilde{B}_t$. Of course, you could integrate over the same variable as in part (a) instead, in which case your sine term would be $\sin(e^u)$.  
