1. (14 pts) The American call option works as follows: it has a strike price $K$, and a deadline $T$. At any one point up to $T$, you’re allowed to buy one unit of stock for price $K$. The American call option doesn’t have a fixed execution time, only a deadline. In this problem, you will analyze the American call option with deadline $T = 2$ in the binomial market.

Let the initial stock price be $S_0 = 100$, with $U = 1.1$ and $D = 0.8$. There is no interest rate: $r = 0$. Our American option will have strike $K = 85$.

(a) Sketch the tree of possible stock prices from time 0 through time 2. Fill in the value of the American option at time 2.

(b) At time 1, you have a choice: you can execute the American option immediately, or you can treat it as a European option that will expire at time 2. Compute the correct value of each of these possibilities. Determine which of them has a higher value: that’s the value of the American option at time 1.

(c) At time 0, you can execute the option immediately, or hold on to the contract that has the two possible values you just computed at time 1. The larger of these is the value of the American option at time 0: what is it?

Solution:

(a) The two possibilities at time 1 are 110 and 80; the three possibilities at time 2 are 121, 88, and 64. The time-2 possible values of the American option are 36, 3, or 0.

(b) If $S_1 = 80$, exercising the option immediately is silly. Waiting for one time unit and exercising then has value

$$p \cdot 3 + (1 - p) \cdot 0,$$

where $p \cdot 1.1 + (1 - p) \cdot 0.8 = 1$

so $p = 2/3$ and the value if $S_1 = 80$ is 2.

If $S_1 = 110$, executing it right now has value 25. The fair value of waiting for one time unit and exercising it then is given by

$$p \cdot 36 + (1 - p) \cdot 3,$$

$p = 2/3 \implies \text{value} = 25$.

This coincides with the value of exercising it right now, so if $S_1 = 110$ then the value is 25.

(c) At time 0, the value of exercising the option immediately is 15, while the value of waiting is given by

$$p \cdot 25 + (1 - p) \cdot 2 = \frac{52}{3} \approx 17.33.$$ 

So the value is 17.33.

In the absence of dividends, it’s never optimal to exercise an American option prematurely, so when there aren’t dividends involved, an American option is the same as a European one.