1. (6 pts) For the pairs of random variables $X$ and $Y$ below, decide whether $Y$ is $\sigma(X)$-measurable, $X$ is $\sigma(Y)$-measurable, both, or neither. You do not have to justify your answer.

(a) $\Omega = \{\text{Alice, Annabelle, Candace, Caroline, Robert, Rhett}\}$. (The first four are female names, the last two are male names.) $\mathbb{P}$ = uniform probability measure. $X : \Omega \rightarrow \{A, \ldots, Z\}$ gives the first letter of the name, $Y : \Omega \rightarrow \{A, \ldots, Z\}$ gives the last letter of the name.

(b) $\Omega = \{\text{permutations of the numbers 1, 2, and 3}\}$, meaning these three numbers written in some order without repetitions. $\mathbb{P}$ is positive on each element of $\Omega$. $X : \Omega \rightarrow \mathbb{R}$ computes (first number) - (second number) + (third number), $Y : \Omega \rightarrow \mathbb{R}$ writes the three numbers as the digits of a three-digit number. That is, $X(2, 1, 3) = 2 - 1 + 3 = 4$ and $Y(2, 1, 3) = 213$.

Solution:

(a) $Y$ is $X$-measurable: if $X = A$ then $Y = e$, if $X = C$ then $Y = e$, and if $X = R$ then $Y = t$. $X$ is not $Y$-measurable, because if $Y = e$, $X$ could be either A or C.

(b) $X$ is $Y$-measurable, since $Y$ uniquely determines the permutation. $Y$ is not $X$-measurable, because 2, 1, 3 and 3, 1, 2 are indistinguishable to $X$.

2. (9 pts) Sketch the CDF of the following measures. Make sure to indicate the values of any of the useful points.

(a) The uniform probability measure on the set $[-1, 1] \cup [3, 5]$.

(b) The measure assigning mass $\frac{1}{4}$ to each of $-1, 1, 3, 5$. (That is, $m(\{i\}) = 1/4$ if $i = -1, 1, 3, 5$, and $m(A) = 0$ if $A \cap \{-1, 1, 3, 5\} = \emptyset$.)

(c) The law of $X$, where $X$ is a real-valued random variable defined as follows. Let $S = \{U, D\}^2$ with the uniform probability measure; $X(s)$ is the number of $D$’s in the sequence.

Solution: