1. Match the following measures on \( \mathbb{R} \) to the plots of their CDFs.

(a) The uniform probability measure on the set \([-3, -2] \cup [1, 5]\).
   \textbf{Answer:} C.

(b) The measure assigning mass \( \frac{1}{4} \) to each of \(-1, 1, 3, 5\). (That is, \( m\{i\} = 1/4 \) if \( i = -1, 1, 3, 5 \), and \( m(A) = 0 \) if \( A \cap \{-1, 1, 3, 5\} = \emptyset \).)
   \textbf{Answer:} B.

(c) The measure which assigns mass 1 to the point 3, and 0 otherwise. Formally, \( m(A) = 1 \) if 3 \( \in \) \( A \), and \( m(A) = 0 \) if 3 \( \not\in \) \( A \).
   \textbf{Answer:} A.

(d) The law of \( X \), where \( X \) is a real-valued random variable defined as follows. Let \( S = \{A, B\}^2 \) with the uniform probability measure (i.e. all one-element sets have the same measure); \( X(s) \) is the index of \( s \) in alphabetical order (so \( X(AA) = 1 \) and \( X(BB) = 4 \)).
   \textbf{Answer:} D.

(e) The law of \( X \), where \( X \) is a real-valued random variable defined as follows. Let \( S = \{U, D\}^2 \) with the uniform probability measure; \( X(s) = 3 \) for all \( s \in S \).
   \textbf{Answer:} A again.

(f) The law of \( X \), where \( X \) is a real-valued random variable defined as follows. Let \( S = \{A, B, C, D\} \) with the uniform measure, and let \( X(A) = 3 \), \( X(B) = -1 \), \( X(C) = 1 \), and \( X(D) = 5 \).
   \textbf{Answer:} B again.

(g) The law of \( X \), where \( X \) is a real-valued random variable defined as follows. Let \( S = [0, 1] \) with the uniform probability measure (i.e. \( m(A) = l(A) \) if \( A \subset S \)), and let \( X(s) = -3s + 2 \).
   \textbf{Answer:} E.

2. (8 pts) For each pair of random variables \((X, Y)\) below, decide whether \( Y \) is \( \sigma(X) \)-measurable, \( X \) is \( \sigma(Y) \)-measurable, both, or neither. \textbf{You do not have to justify your answer.}

(a) \( S = \{A, B, C\}^2 \) with the uniform probability measure (i.e. each sequence has measure \( 1/9 \)). \( X : S \to \mathbb{R} \) is the index of the sequence in alphabetical order, \( Y : S \to \{A, B, C\} \) is the first letter of the sequence.
**Answer:** $Y$ is $\sigma(X)$-measurable, since for each index there’s only one first letter. $X$ is not $\sigma(Y)$-measurable, since here are multiple indices corresponding to a single first letter.

(b) $S = \mathbb{Z}_+$ with the probability measure $P(n) = e^{-\lambda}\lambda^n/n!$, $n \geq 0$. $X : S \to \mathbb{R}$ is given by $X(n) = n^3$, $Y : S \to \mathbb{R}$ is given by $Y(n) = n$.

**Answer:** The probability measure on $S$ doesn’t matter at all for this question. $X$ is $\sigma(Y)$-measurable, and $Y$ is $\sigma(X)$-measurable, since given $n$ I can find $n^3$ and given $n^3$ I can find $n$.

(c) $S = \mathbb{Z}$ with the probability measure given by $P(0) = 0$, $P(\pm n) = \frac{1}{2}e^{-\lambda}\lambda^{n-1}/(|n| - 1)!$, i.e. roughly half of the measure from above. $X : S \to \mathbb{R}$ is given by $X(n) = n^3$, $Y : S \to \mathbb{R}$ is given by $Y(n) = n$.

**Answer:** The probability measure on $S$ still doesn’t matter. The random variables are still measurable in both directions, as there’s exactly the same information contained in the value of $X$ as in the value of $Y$.

(d) $S = [0, 1] \times [0, 5]$ with the uniform measure, and for $s = (a, b) \in S$, $X(s) = a$, $Y(s) = b$.

**Answer:** Not measurable in either direction. Knowing the value of $X$ tells me nothing about the value of $Y$, and knowing the value of $Y$ tells me nothing about the value of $X$. 