FM 5011: Mathematical Background for Finance I  
Quiz 3 solutions

1. (3 pts) Let $S = [0, 2]$ equipped with the uniform probability measure (and the Borel $\sigma$-algebra).  
   Careful: $S \neq [0, 1]$!
   (a) Compute $E[X]$, where $X : S \to \mathbb{R}$ is given by $X(s) = s^2$.
   (b) Compute $E[Y]$, where $Y : S \to \mathbb{R}$ is given by
   
   $$Y(s) = \begin{cases} 
   -1, & x < 1 \\
   1, & x \geq 1. 
   \end{cases}$$

   Solution:
   (a) Notice that the probability distribution on $S$ is $1/2$ times the Lebesgue measure. In particular, when we integrate with respect to it, we should be integrating with respect to $ds/2$.
   
   $$E[X] = \int_0^2 s^2 \frac{ds}{2} = \frac{8}{6} = \frac{4}{3}.$$  
   This makes intuitive sense: the average value of $s^2$ on the interval from 0 to 2 should be between 0 and 2, and if you think about it, it should be bigger than 1 (because the deviations in the positive direction are larger than in the negative direction).
   (b) $Y$ is a discrete random variable: $P(Y = -1) = \frac{1}{2} = P(Y = 1)$. Thus, $E[Y] = 0$. You could also get this by symmetry, or by integration:
   
   $$E[Y] = \int_0^1 (-1) \cdot \frac{1}{2} ds + \int_1^2 (-1) \cdot \frac{1}{2} ds = 0.$$  

2. (5 pts) For each pair of probability measures $\mu, \nu$ on $S$ below, decide whether $\mu \ll \nu$, $\nu \ll \mu$, both (i.e. $\mu \sim \nu$), or neither.
   (a) (1 pt) Let $S = \{U, D\}^8$, with the $\sigma$-algebra $F = 2^S$. Let $\mu$ be the product measure that corresponds to taking $P(U) = 0.8, P(D) = 0.2$ and let $\nu$ be the product measure that corresponds to taking $P(U) = 0.3, P(D) = 0.7$.
   Solution: The only set of $\mu$-measure 0 is the empty set; this is also the only set of $\nu$-measure zero. Since the collection of sets of measure 0 is the same for the two measures, they are equivalent: $\mu \sim \nu$, i.e. both $\mu \ll \nu$ and $\nu \ll \mu$.
   (b) (2 pts) Let $S = \mathbb{R}$, let $\mu$ be the distribution of $X$, and let $\nu$ be the distribution of $Y$, where the CDFs of $X$ and $Y$ are plotted below.

   Solution: Because there are intervals where $F_Y$ is flat but $F_X$ isn’t, $\mu \not\ll \nu$. (There are some values that $X$ can take that $Y$ cannot take. In particular, the smallest values of $X$ are smaller than the smallest values of $Y$, and the largest values are larger.) On the other hand, $X$ and $Y$ both have
densities, and the density of $X$ is non-zero everywhere the density of $Y$ is non-zero. This means $d\nu/d\mu$ is well-defined, so $\nu \ll \mu$.

(c) (2 pts) Let $S = \{\pm 2, \pm 1\}^2$, and let the $\sigma$-algebra $\mathcal{F}$ be generated by the sets $A = \{\text{both numbers are negative}\}$, $B = \{\text{both numbers are positive}\}$. The measures are given by $\mu(A) = 1/3$, $\mu(B) = 1/3$ and $\nu(A) = 1/2$, $\nu(B) = 1/2$. (Remember that these are probability measures!) In addition to deciding whether $\mu \ll \nu$, $\nu \ll \mu$, both, or neither, decide how many sets there are in $\mathcal{F}$.

Solution: I’ll start with the number of sets in $\mathcal{F}$. We can write $S = A \cup B \cup (A \cup B)^c$. The complement is the set “exactly one coordinate is negative”. These three sets are all in $\mathcal{F}$, they are disjoint, their union is $S$, and they are the smallest measurable sets. So everything in $\mathcal{F}$ will be a union of some of these, and there are $2^3 = 8$ measurable sets in total.

Now, $\mu$ assigns measure $1/3$ to each of the three basic sets in $\mathcal{F}$, and $\nu$ assigns measure $0$ to $(A \cup B)^c$. This means that $\nu \ll \mu$ (because the only measurable set with $\mu$-measure $0$ is the empty set), but $\mu \not\ll \nu$ (because the set “one coordinate is negative” can occur under $\mu$ but not under $\nu$).