1. (9 pts) For each pair of events $A, B$ below, determine whether they are independent. Justify your answer briefly.

(a) Consider the binomial market with $S_0 = 100$, $U = 1.1$, $D = 0.9$, $\mathbb{P}(U) = 0.7$, and $r = 0.05$. Let $A$ be the event that $S_1 = 110$. Let $B$ be the event that $S_2 = 81$.

(b) Let $A$ be the even that when you roll two fair 6-sided dice, the sum is $\geq 7$. Let $B$ be the event that when you roll the first of the two dice, you get $\geq 4$.

(c) Let $A$ and $B$ be such that $\mathbb{P}(A) = \mathbb{P}(B) = \frac{2}{3}$ and $\mathbb{P}(A \cup B) = \frac{8}{9}$.

Solution:

(a) Not independent: $\mathbb{P}(A \cap B) = 0 \neq \mathbb{P}(A)\mathbb{P}(B)$.

(b) Not independent: $\mathbb{P}(A) = \frac{1}{2}$, but $\mathbb{P}(A|B) > \frac{1}{2}$, since for each value of the first die, more than half of the possibilities of the second die give you a sum of at least 7. You could also compute $\mathbb{P}(A \cap B) = \frac{1}{6} \cdot \frac{4}{6} + \frac{1}{6} \cdot \frac{5}{6} + \frac{1}{6} \cdot 1 = \frac{5}{12} \neq \frac{1}{4}$.

(c) Independent: $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = \frac{4}{9}$. (The intersection is exactly what you’re double-counting when you add up $\mathbb{P}(A) + \mathbb{P}(B)$).

2. (5 pts) Find the covariance of the random variables whose joint CDF is given by

$$\mathbb{P}(X \leq a \text{ and } Y \leq b) = ab(2a^2 - 3ab + 2b^2), \quad 0 \leq a \leq 1, \quad 0 \leq b \leq 1.$$ Outside the square, $\mathbb{P}(X \leq 0) = \mathbb{P}(Y \leq 0) = \mathbb{P}(X > 1) = \mathbb{P}(Y > 1) = 0$.

Hint: if you want the joint density, differentiate with respect to both $a$ and $b$.

Solution:

$$f_{X,Y}(a,b) = \frac{d^2}{dadb}F_{X,Y}(a,b) = 6a^2 + 6b^2 - 12ab = 6(a-b)^2.$$ The individual densities are

$$f_X(a) = \int_0^1 f_{X,Y}(a,b)db = \int_0^1 6(a-b)^2db = 6a^2 - 6a + 2$$

and similarly for $f_Y$ (the two variables are obviously symmetric).

Consequently,

$$\mathbb{E}[X] = \mathbb{E}[Y] = \int_0^1 (6a^3 - 6a^2 + 2a)da = \frac{1}{2},$$

and

$$\mathbb{E}[XY] = \int_0^1 \int_0^1 6ab(a-b)^2dadb = \frac{1}{6}.$$ The covariance is

$$\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{1}{6} - \frac{1}{4} = -\frac{1}{12}.$$ Note that you could get the mean of $X$ and $Y$ as $1/2$ from noticing that the distribution is symmetric about the point $(\frac{1}{2}, \frac{1}{2})$. You can also check that the answer should be negative from the fact that the density is highest when $x - y$ is large, and lowest when $x = y$; so you expect $x$ and $y$ to be anticorrelated.