1. (5 pts) Let \((X, Y)\) be jointly normal with mean \((1, -2)\) and covariance \(\Sigma = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\). Find \(\mathbb{E}[3X + Y]\), \(\text{Var}(3X + Y)\), and write down the density of \(3X + Y\).

**Solution:** \(\mathbb{E}[3X + Y] = 3 - 2 = 1\), \(\text{Var}(3X + Y) = 9 \text{Var}(X) + \text{Var}(Y) + 6 \text{Cov}(X, Y) = 14\). Since \((X, Y)\) is jointly normal, \(3X + Y\) is normal, and so its density is

\[
f_{3X+Y}(a) = \frac{1}{\sqrt{2\pi}\sqrt{14}} \exp\left(-\frac{1}{2} \frac{(a - 1)^2}{14}\right).
\]

2. (6 pts) Let \((B_1, B_2)\) be jointly normal with mean 0 and covariance matrix

\[
\Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.
\]

Write down the joint density of \((B_1, B_2)\):

\[
f_{(B_1,B_2)}(a, b) =
\]

To obtain full credit, you should expand any matrix–vector products in your answer.

**Solution:** The determinant is \(\det \Sigma = 2 - 1 = 1\) and \(\Sigma^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\), so the joint density is

\[
f_{(B_1,B_2)}(a, b) \frac{1}{2\pi} \exp\left(-\frac{1}{2} (2a^2 + b^2 - 2ab)\right).
\]

3. (4 points) Let \(X\) be a standard normal random variable. Let \(Y = 3 + 2X\), and let \(Z = -2 - X\). Are \(Y\) and \(Z\) jointly normal? Find \(\text{Cov}(Y, Z)\). Are \(Y\) and \(Z\) independent?

**Solution:** \((Y, Z)\) are jointly normal, being obtained as a linear transformation of \(X\). We have

\[
\text{Cov}(Y, Z) = \text{Cov}(3 + 2X, -2 - X) = \text{Cov}(2X, -X) = -2 \text{Var}(X) = -2.
\]

\(Y\) and \(Z\) are not independent, since their covariance is not 0 (and also since \(Y = -2Z - 1\) is \(\sigma(Z)\)-measurable and non-constant).