FM 5011: Mathematical Background for Finance I
Quiz 6, October 29, solutions

1. (10 pts) Let $B_t, t \geq 0$ be a Brownian motion. Let $X = B_1 + B_2$, and let $Y = B_1 + B_3$.

(a) Are $X$ and $Y$ independent?
(b) Are $X$ and $Y - X$ independent?

Hint: compute the covariances. Justify your answer (including why covariance is useful, if you’re using it).

Solution: Since any finite number of $B_t$’s are jointly normal, we see that $X, Y, Y - X$, etc. are also all jointly normal. Consequently, they’re independent if and only if covariances are zero. (This is only true because we have joint normality!)

(a) $\text{Cov}(X, Y) = \text{Cov}(B_1 + B_2, B_1 + B_3) = \text{Var}(B_1) + \text{Cov}(B_1, B_2) + \text{Cov}(B_1, B_3) + \text{Cov}(B_2, B_3) = 5$.

No, they’re not independent.

(b) $Y - X = B_3 - B_2$. This is independent of anything that happens up to time 2, in particular it’s independent of $B_1 + B_2$. So yes, they are independent. (You could also compute the covariance and find that it’s zero.)

2. (7 pts) Let $B_t, t \geq 0$ be a Brownian motion, and let $X = B_1$, $Y = B_2$. By the independent increments property of Brownian motion, we know that $Y - X$ is independent of $X$. Is it also independent of $Y$?

Solution: No, $\text{Cov}(B_2 - B_1, B_2) = \text{Var}(B_2) - \text{Cov}(B_1, B_2) = 2 - 1 = 1$. They’re not independent.

Another way of thinking about it is that $B_2 = (B_2 - B_1) + (B_1)$. The two summands are independent of each other, but the sum depends on each of them.

3. (10 pts) Let $B_t, t \geq 0$ be a Brownian motion. Find $\mathbb{E}[B_t^4]$. You may find the following integral useful:

$$\int_{-\infty}^{\infty} x^4 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 3.$$ 

Solution: The integral gives you $\mathbb{E}[Z^4]$ where $Z \sim N(0, 1)$. We know that $B_t \sim N(0, t)$, i.e. $B_t = \sqrt{t} \cdot N(0, 1)$. Consequently,

$$\mathbb{E}[B_t^4] = (\sqrt{t})^4 \cdot \mathbb{E}[N(0, 1)^4] = 3t^2.$$