Important concepts from today: arbitrage; hedging; European option; risk-neutral measure and risk-neutral valuation.

1. Binomial market model

Consider a market that has one type of stock and one type of riskless bond. The stock price at time $t$ is $S_t$; the bond price at time $t$ is $B_t$. We will assume that time is discrete, e.g. measured in days.

Bonds are a completely known quantity: we have

$$B_{t+1} = (1 + r)B_t \implies B_t = (1 + r)^t B_0$$

Here, $r$ is the interest rate.

Stock prices, on the other hand, may fluctuate. Let us assume that $S_{t+1} = \begin{cases} US_t, & \text{if stock price goes up} \\ DS_t, & \text{if stock price goes down} \end{cases}$, where $D < (1 + r) < U$.

**Fundamental question:** how should we price financial derivatives in this market?

**Definition 1.** For the time being, a financial derivative is a product which will pay some amount of money at time $T$; the amount depends on stock prices up to and including $T$.

We may later encounter financial derivatives that do not have a fixed execution time, at which point we will need to be more careful about what the payout will be.

**Example 1** (European options). The two most important financial derivatives for this course are the European options.

A European call option gives the holder an option, but not the obligation to buy one unit of the stock at time $T$ for a price $K$. Therefore, its payout at time $T$ is $(S_T - K)_+ = \max(S_T - K, 0)$.

A European put option gives the holder an option, but not the obligation to sell one unit of the stock at time $T$ for a price $K$. Therefore, its payout at time $T$ is $(K - S_T)_+ = \max(K - S_T, 0)$.

The price $K$ is called the strike.

We assume that the stocks, bonds, and derivatives are freely traded on the market. This means that at any time, you can hold $x$ units of stocks, or bonds, or derivatives, for any real value of $x$ (positive, negative, non-integer). Negative values of $x$ correspond to short-selling the corresponding commodity: we assume there’s always a buyer for that.

**Definition 2.** A portfolio is a combination of stocks, bonds, and financial derivatives. The value of the portfolio at time $t$ is the price of all the items in it at time $t$. 

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Definition 3 (Arbitrage). If there is some portfolio that is initially worth 0 (no investment) and is guaranteed to be worth at least $\epsilon > 0$ at some fixed future time $t$, we say there’s arbitrage in the market. If there is a fixed time $t$ when the portfolio is guaranteed to be worth $\geq 0$, and has a positive probability of being worth $> 0$, we say there’s statistical arbitrage.

Arbitrage means that, without any risk, you can make an unbounded amount of money with no initial investment. We’ll figure out prices in a market where this doesn’t happen: if the real-world price doesn’t match our computation, we’ll see that we can make a riskless profit from trading in the market.

Note: If there is no arbitrage in the market, then it’s impossible for a portfolio to be a guaranteed loss of money too: otherwise you could make profit from selling it.

2. Hedging

It’s easy to compute the price of a portfolio that consists only of bonds and stocks: if I have $a$ stocks and $b$ bonds, then the portfolio is worth $aS_t + bB_t$. Notice that if all portfolios have only stocks and bonds in them, then there is no arbitrage: a portfolio that starts out with $a$ stocks and has a net worth of 0 has $b = -(aS_0)/B_0$ bonds. If the stock price always goes up or always goes down, the portfolio will be worth

$$V_{\text{max}} = a(U^t - (1 + r)^t)S_0 > 0, \quad V_{\text{min}} = a(D^t - (1 + r)^t)S_0 < 0.$$  

So there’s no riskless profit in the market with only stocks and bonds.

Now let’s introduce a financial derivative, and let’s try to replicate it using stocks and bonds. For example, if our financial derivative will pay out an amount $\Phi(S_1)$ at time 1, let’s find a combination of stocks and bonds that produces the same payout:

$$\Phi(S_1) = aS_1 + bB_1$$

Clearly, we can’t do this for just any function $\Phi$, since not all functions are linear. However, we know that $S_1$ will take one of two values, so we only need to match

$$\begin{cases} 
\Phi(US_0) = aUS_0 + bB_1, \\
\Phi(DS_0) = aDS_0 + bB_1.
\end{cases}$$

This we can solve for $a$ and $b$. This means that our financial derivative behaves exactly like a combination of $a$ stocks and $b$ bonds. Consequently, it should be worth the same amount even at time 0, i.e. the value of the financial derivative $\Phi(S_1)$ at time 0 should be

$$aS_0 + bB_0, \quad \text{for the } a \text{ and } b \text{ that solve (1)}.$$  

Exercise 1 (In-class exercise). Suppose the initial stock price is $S_0 = 100$, $S_1$ is either 120 (with probability 0.6) or 80 (with probability 0.4), the strike is $K = 110$, and for simplicity take $r = 0$ (no interest). Compute the value of the European call $C(K, 1)$ at time 0.

Solution 1. We have $U = 1.2, \ D = 0.8$; if the stock goes up, then $\Phi(US_0) = (120 - 110)_+ = 10$, and if the stock goes down, then $\Phi(DS_0) = (80 - 110)_+ = 0$. Solving the system of equations

$$b = \frac{1.2 \cdot 0 - 0.8 \cdot 10}{1.2 - 0.8} = -20, \quad a = \frac{10 - 0}{120 - 80} = \frac{1}{4}$$

and the price is

$$-20 + \frac{1}{4} \cdot 100 = \$5.$$
Note: We did not need to use probability to derive this answer. For the purposes of pricing derivatives, it does not matter how likely the stock is to go up or down. What does matter is the set of possibilities for the future values of the stock. Intuitively, the expected value of the stock price change should influence the price of the stock; financial derivatives care about the fluctuations around that drift.

Note 2: To solve the system of equations (1), we needed to know that there were exactly two possible values for $S_1$: our argument wouldn’t have worked with more than 2 possibilities. If we were doing this in a market with more than two instruments, we would have a dimension-counting argument: the number of possibilities equations (i.e. the number of probabilistic outcomes) needs to match the number of unknowns that we’re solving for.

Question: How do we make money if someone in the market is offering to trade this call option for 6? We know that it’s worth 5, so we should clearly sell him many options, but what do we do next?

Answer: Sell one call option for 6. Set 1 aside. Replicate the option according to the recipe above: borrow $20 units from the bank (“short-sell $20 worth of the bond”) and use them and the extra $5 from the option to buy 1/4 of a stock.

If the stock goes up, your counterparty will call on you to deliver one unit of stock for $K = 110$. You already have 1/4 units of stock, so you need to buy another 3/4 units. This will cost you another $90$. Thus, $K = 110$ is exactly enough to repay your bank loan, and you have just made $1$.

If the stock goes down, your counterparty won’t exercise the option. Sell your 1/4 of a stock, and repay the bank loan. You again have made $1$.

Thus, you earn $1$ for each call option you sell, regardless of what happens to the stock price. This lets you make an arbitrarily large profit!

We now know how to go from the definitive payout $\Phi(T)$ to the fair value of the derivative at time $(T-1)$. This allows us to work backwards: starting from the possible values at $(T-1)$, find the values at time $(T-2)$, and so on.

Example 2 (Many-period European call). Let $S_0 = 80$, $U = 1.5$, $D = 0.5$, $r = 0$. Consider the European call $C(80,3)$ with strike price $K = 80$ and deadline $T = 3$. Then we have a binary tree of possibilities, and as a group exercise in class we solved for the replicating portfolios at all times, and determined the value of the call at all times.

For example, to find the price when $S_1 = 120$, we looked at the values at time 2, and solved

$$\begin{align*}
    x + y \cdot 180 &= 100 \\
    x + y \cdot 60 &= 5
\end{align*}$$

and then computed $x + y \cdot 120$.

Exercise 2 (In-class exercise). I am offering to buy or sell the option from the binary tree for $25$. Show how to make sure money from my offer. Explain in particular what your actions will be at each of $t = 0, 1, 2, 3$ if the stock price trajectory is $S_t = 80, 120, 60, 30$.

Solution 2. Because 25 is less than the true value (27.5), you should buy the option. You should then replicate $(-1)$ times the option. That is:

(0) At time 0, you have borrow $25$ from the bank to buy the option. You want your portfolio to be $22.5 - 5/8S_0$, so you set $2.50$ aside, and short-sell (“borrow”) $5/8$ of a share.
Figure 1. The numbers inside the ovals are possible stock prices. The numbers inside rectangles are the values of the call at the corresponding time. $x$ and $y$ give the replicating strategy: to replicate the call for one time period, you should hold $\$x$ worth of bond, and $y$ units of the stock.

1. At time 1, the replicating portfolio says you should have $42.5 - \frac{95}{120}S_1$. Since you’ve already short-sold $\frac{5}{8} = \frac{75}{120}$ of a share, you need to short-sell another $\frac{20}{120} = \frac{1}{6}$ of a share (for $\$20$).

2. At time 2, your position should be $5 - \frac{1}{6}S_2$. So you use $\$37.5$ to buy $\frac{5}{8}$ of a share.

3. At time 3, since $S_t = 30$, there is no point in exercising the option you bought. Instead, you should settle your accounts by using the $\$5$ you have left to buy the remaining $\frac{1}{6}$ of a share. You have made a profit of $\$2.50$.

3. Risk-neutral valuation: an easier way to compute value

If you look at the binary tree, you might notice a pattern: each value is the average of the two values to the right of it. This would have given us a much easier way to compute the value of the call, had we known it in advance!

We know the no-arbitrage price of the derivative doesn’t depend on the probabilities of the stock going up and down. Now, suppose $p$ is such that

$$pU + (1 - p)D = (1 + r).$$

This is not the true probability of the stock going up or down, but if it were, then

$$\mathbb{E}_p[S_{t+1} | S_t] = pUS_t + (1 - p)DS_t = (1 + r)S_t.$$ 

In particular, if I have a combination of stocks and bonds that’s initially worth 0, then at any later time,

$$\mathbb{E}_p[S_t + B_t] = (1 + r)\mathbb{E}_p[S_{t-1} + B_{t-1}] = \ldots = (1 + r)^t\mathbb{E}_p[S_0 + B_0] = 0.$$

Here, I’m repeatedly using the conditional expectation property

$$\mathbb{E}[S_t] = \mathbb{E}[\mathbb{E}(S_t | S_{t-1})] = \mathbb{E}[(1 + r)S_{t-1}] = (1 + r)\mathbb{E}[S_{t-1}].$$

Since we can clearly solve for $p$, this gives us another way of showing that there is no arbitrage in the stock-and-bond market: we found some way to take the weighted average
of all the possible portfolio prices that gives us 0 in expectation, and therefore there can’t be riskless profit in the market.

Now, suppose we know that it’s possible to replicate our financial derivative \( \Phi(S_T) \) using stocks and bonds somehow. The argument we just gave tells us that, with respect to the averaging induced by \( p \), the average value of the portfolio always grows by a factor of \((1 + r)\) per time period. Consequently,

\[
V_0 = (1 + r)^{-T} \mathbb{E}_p[\Phi(S_T)], \quad V_t = (1 + r)^{t-T} \mathbb{E}_p[\Phi(S_T)].
\]

That is, to find the fair price of the option, you figure out all the possible prices \( S_T \) and associated payouts \( \Phi(S_T) \), and you compute their weighted average using the risk-neutral measure \( p \) instead of the true probabilities.

For example, for the European call, \( p = 1/2 \), so \( V_{t-1} = \frac{1}{2} V_t(\text{up}) + \frac{1}{2} V_t(\text{down}) \). We can also compute

\[
V_0 = p^3 \Phi(270) + 3p^2(1-p)\Phi(90) + 3p(1-p)^2\Phi(30) + (1-p)^3\Phi(10)
= \frac{1}{8} \cdot 190 + \frac{3}{8} \cdot 10 + \frac{3}{8} \cdot 0 + \frac{1}{8} \cdot 0 = 27.50.
\]

Make sure you understand why it’s \( 3p^2(1-p) \) and not just \( p^2(1-p) \)!

Exercise 3 (In-class exercise). Suppose \( T = 5 \), so that there are 6 possible outcomes in the binary tree. Let the pay-off for them be given by 108, 81, 54, 27, 18, 9. Suppose also that \( U = 1.6 \), \( D = 0.7 \), and \( r = 0 \). What’s the value of this contract at time 0?

Solution 3. First solve for \( p \): \( p = 1/3 \).

Next,

\[
V_0 = p^5 \cdot 108 + 5p^4(1-p) \cdot 81 + \binom{5}{2} p^3(1-p^2) \cdot 54 + \binom{5}{3} p^2(1-p)^3 \cdot 27
+ 5p(1-p)^4 \cdot 18 + (1-p)^5 \cdot 9.
\]

We don’t actually care very much about the number here.

Notice that if we want to find the replicating portfolio, we can now do it as we go along, without working out all the systems of equations along the entire binary tree.

Remark 1 (Risk-neutral measure is not the truth!). The risk-neutral measure lets us compute values of financial derivatives as if all financial products had the same expected profit. It is just a computational device! We do not actually believe that all products have the same expected profit.

Main goal of this course: To do a similar analysis in continuous time. The main consternation is how to replace the argument that involved \( \mathbb{E}(S_t|S_{t-1}) \).