Important concepts from today: Itô lemma for more involved functions, $dBdt = 0$, Black-Scholes pricing of a European call by a lot of differentiation, Black-Scholes pricing of a European call by a risk-neutral measure.

1. Review of last time

We defined the martingale transform: if $X_n$ is a martingale with respect to filtration $\mathcal{F}$, and $C_n$ is predictable, meaning $C_n$ is $\mathcal{F}_{n-1}$-measurable, then

$$Y_n = \sum_{k=1}^{n} C_k(X_k - X_{k-1})$$

is a martingale also adapted to $\mathcal{F}$. (We need an additional condition, e.g. $\mathbb{E}[X_n^2] < \infty$, $\mathbb{E}[C_n^2] < \infty$; we won’t worry about it.)

Taking this concept into continuous time, we defined the stochastic integral,

$$\int_0^t f(s)dB_s,$$

where $f$ is predictable, that is, measurable with respect to $\sigma(B_u, u \leq t)$. (And also $\int_0^t \mathbb{E}[f(s)^2]ds < \infty$.) We showed that this is different from the ordinary Riemann integral; for example,

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t,$$

and not $\frac{1}{2}B_t^2$.

We showed some properties of the stochastic integral:

1. The stochastic integral $X_t = \int_0^t f(s)dB_s$ is a martingale. In particular, $\mathbb{E}[X_t] = \mathbb{E}[X_0] = 0$. A good way to remember this is that expectation commutes with integral, and the predictable function $f$ is independent of the Brownian increment $dB$, so

$$\mathbb{E}[\int_0^t f(s)dB_s] = \int_0^t \mathbb{E}[f(s)dB_s] = \int_0^t \mathbb{E}[f(s)]\mathbb{E}[dB_s] = 0.$$ 

2. Integral is linear in the integrand: $\int_a^b (f(s) + g(s))dB_s = \int_a^b f(s)dB_s + \int_a^b g(s)dB_s$

3. Also linear in the domain: $\int_a^b f(s)dB_s + \int_b^c f(s)dB_s = \int_a^c f(s)dB_s$

4. The Itô isometry lets us compute the variance of the integral:

$$\text{Var} \left( \int_0^t f(s)dB_s \right) = \mathbb{E}[\left( \int_0^t f(s)dB_s \right)^2] = \int_0^t \mathbb{E}[f(s)^2]ds.$$ 

Notice that on the right we have an ordinary Riemann integral of a non-random function, and we’ve assumed it’s finite.

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(5) The integral is continuous, i.e. has continuous sample paths.
We derived the fundamental relationships
\[ dt^2 = 0, \quad dB_t^2 = dt, \quad dBdt = 0 \]
and used them to show that for stochastic calculus,
\[
d(f(B_t)) = f'(B_t)dB_t + \frac{1}{2} f''(t)dt, \quad f(B_t) - f(B_0) = \int_0^t f'(B_s)dB_s + \frac{1}{2} \int_0^t f''(B_s)ds.
\]
and
\[
df(t, B_t) = \partial_B f(t, B_t)dB + \left( \partial_t f(t, B_t) + \frac{1}{2} \partial_{BB} f(t, B_t) \right) dt,
\]
or
\[
f(t, B_t) - f(0, B_0) = \int_0^t \partial_B f(s, B_s)ds + \int_0^t \left( \partial_s f(s, B_s) + \frac{1}{2} \partial_{BB} f(s, B_s) \right) ds.
\]
The way to show this was to use Taylor’s expansion and keep writing until terms start vanishing, so for example,
\[
f(t + dt, B + dB_t) = f(t, B_t) + dt \frac{\partial}{\partial t} f(t, B_t) + dB_t \frac{\partial}{\partial B_t} f(t, B_t)
\]
\[+ \frac{1}{2} \frac{dt^2}{=0} \frac{\partial^2}{\partial t^2} f(t, B_t) + \frac{1}{2} \frac{dB_t^2}{=dt} \frac{\partial^2}{\partial B_t^2} f(t, B_t)
\]
\[+ \frac{dt}{dtdB_t} \frac{\partial}{\partial dB_t} f(t, B_t) + \frac{dtdB_t}{=0} \frac{\partial^2}{\partial dB_t^2} f(t, B_t) + \ldots\]
\[\approx f(t, B_t) + \partial_B f(t, B_t)dB + \left( \partial_t f(t, B_t) + \frac{1}{2} \partial_{BB} f(t, B_t) \right) dt.
\]

Reminder: When you’re taking partial derivatives, you get to pretend that variables don’t depend on each other. That is, \( f \) is just a function of two variables that have nothing to do with each other, and you’re taking the partial derivative of \( f \) with respect to its first input or its second input. You just happen to be evaluating it at a point where \( B_t \) depends on \( t \).

Example 1. Determine whether \( f(t, B_t) = e^{-t/2+B_t} \) is a martingale.

Solution: Differentiate:
\[
df(t, B_t) = -\frac{1}{2} e^{-t/2+B_t} dt + e^{-t/2+B_t} dB_t + \frac{1}{2} e^{-t/2+B_t} dB_t dt = e^{-t/2+B_t} dB_t.
\]
Consequently, \( f(t, B_t) = f(0, B_0) + \int_0^t e^{-s/2+B_s} dB_s \) is a stochastic integral, and thus a martingale.

For the particular function \( f(t, B_t) = e^{-t/2+B_t} \) that we just looked at,
\[
df(t, B_t) = f(t, B_t)dB_t.
\]
Ordinarily, this is the equation satisfied by the exponential function: \( f'(x) = f(x) \) means \( f(x) = Ce^x \). In the stochastic world, the solution to the stochastic differential equation \( df(t, B_t) = f(t, B_t)dB_t \) is \( f(t, B_t) = Ce^{-t/2+B_t} \). This is sometimes called the Itô exponential.

Example 2 (Geometric Brownian motion). Let \( X_t = f(t, B_t) = \exp((c - \frac{1}{2}\sigma^2) t + \sigma B_t) \). Then:
\[
dX_t = \sigma X_t dB_t + (c - \frac{1}{2}\sigma^2) X_t dt + \frac{1}{2} \sigma^2 X_t dt = \sigma X_t dB_t + c X_t dt.
\]
That is, $S$ satisfies the stochastic differential equation

$$dX_t = cX_t dt + \sigma X_t dB_t, \quad X_t - X_0 = c \int_0^t X_s ds + \sigma \int_0^t X_s dB_s.$$ 

2. More general Itô rules

Generally, prices of portfolios and contracts in a financial market will satisfy some stochastic differential equations, and you may want to relate them to each other. Suppose

$$dX_t = A_1(t) dt + A_2(t) dB_t, \quad \text{which really means} \quad X_t = X_0 + \int_0^t A_1(s) ds + \int_0^t A_2(s) dB_s.$$ 

Let’s try to differentiate a function $f(t, X_t)$. The rule is the same as always: take the second-order Taylor expansion, but only keep second-order terms if $dB^2$ appears in them (because $dt^2 = dB dt = 0$):

$$df(X_t) = \partial_t f(t, X_t) dt + \partial_X f(t, X_t) dX_t + \frac{1}{2} \partial_{XX} f(t, X_t) (dX_t)^2 = A_2(t)^2 dB_t^2 = A_2(t)^2 dt$$

$$= \left( \partial_t f(t, X_t) + \partial_X f(t, X_t) A_1(t) + \frac{1}{2} \partial_{XX} f(t, X_t) A_2(t)^2 \right) dt + (\partial_X f(t, X_t) A_2(t)) dB_t.$$ 

If we wanted to write this in integral form, we would write

$$X_t - X_0 = \int_0^t A_1(s) ds + \int_0^t A_2(s) dB_s \implies$$

$$f(X_t) - f(X_0) = \int_0^t \left( \partial_s f(s, X_s) + \partial_X f(s, X_s) A_1(s) + \frac{1}{2} \partial_{XX} f(s, X_s) A_2(s)^2 \right) ds$$

$$+ \int_0^t \partial_X f(s, X_s) A_2(s) dB_s.$$ 

Notice that while the $dt$ term gets quite complicated, the $dB$ term changes in a very simple way. This is sometimes handy.

We may also want to differentiate functions of two (or more) solutions of SDEs, like a function of two different call options. Suppose

$$dX_t = A_1(t) dt + A_2(t) dB_t, \quad dY_t = A_3(t) dt + A_4(t) dB_t.$$
Consider $f(t, X_t, Y_t)$: we again do a second-order Taylor expansion, and then simplify terms, using the equations $dt^2 = 0$, $dBdt = 0$, $dB^2 = dt$:

$$
df(t, X_t, Y_t) = \partial_t(f)dt + \partial_X(f)dX_t + \partial_Y(f)dY_t$$

$$+ \frac{1}{2}\partial_{XX}(f)(dX_t)^2 + \frac{1}{2}\partial_{YY}(f)(dY_t)^2 + \underbrace{\partial_{XY}(f)}_{\text{this is } \frac{1}{2}(\partial_{XX} + \partial_{YY})} dX_t dY_t$$

$$= \partial_t(f)dt + \partial_X(f)(A_1(t)dt + A_2(t)dB_t) + \partial_Y(f)(A_3(t)dt + A_4(t)dB_t)$$

$$+ \frac{1}{2}\partial_{XX}(f)(A_2(t)^2dt) + \frac{1}{2}\partial_{YY}(f)(A_4(t)^2dt)$$

$$+ \underbrace{\frac{1}{2}\partial_{XY}(f) A_2(t)A_4(t)dt}_{\text{is } \frac{1}{2}(\partial_{XX} + \partial_{YY})}$$


Example 3. Consider the product of two stochastic processes:

$$d(X_tY_t) = Y_t dX_t + X_t dY_t + dX_t dY_t$$

$$= (Y_tA_1(t) + X_tA_3(t) + A_2(t)A_4(t)) dt + (X_tA_2(t) + Y_tA_4(t)) dB_t$$

The $dX_t dY_t$ the stochastic calculus correction to the classical product rule.

There’s one more generalization to the Itô lemma that we haven’t dealt with yet: what happens when you have multiple Brownian motions around. For completeness, let’s state:

$$dB_t dB_t = 0 \quad \text{when } B_t \text{ and } W_t \text{ are independent Brownian motions.}$$

(If they aren’t independent, replace one of them by the sum of a measurable component and an independent one.)

One more result about stochastic integrals for future reference:

Theorem 2.1 (Martingale representation theorem). Any martingale $M_t$ that is adapted to the Brownian filtration (and is sufficiently integrable) can be represented as

$$M_t = \int_0^t f_s dB_s$$

for some predictable (but possibly random) function $f$ of the Brownian trajectory. This $f$ is (essentially) unique: if the process has two representations

$$M_t = \int_0^t f_s dB_s = \int_0^t g_s dB_s$$

then $f_s = g_s$ for all $s$ with probability 1.

Moreover, if a process $X_t$ (not necessarily a martingale now) has a representation

$$X_t = \int_0^t f_1(s) ds + \int_0^t f_2(s) dB_s = \int_0^t g_1(s) ds + \int_0^t g_2(s) dB_s,$$

then $f_1 = g_1$ and $f_2 = g_2$: the ordinary and the stochastic integrands are each uniquely defined.
3. Black-Scholes option pricing

Let’s put the stochastic calculus theory to use. The Black-Scholes market has one stock; the price of a stock is given by a geometric Brownian motion:

\[ S_t = f(t, B_t) = X_0 e^{(c-\frac{1}{2} \sigma^2)t+\sigma B_t}, \quad \text{or } dS_t = cS_t dt + \sigma S_t dB_t. \]

Here, \( c \) is the mean rate of return, and \( \sigma \) is the volatility or the stock. It determines how much the actual instantaneous rate of return differs from the mean rate of return.

We also have a non-risky asset on the market, with price \( \beta_t = \beta_0 e^{rt} \). Here, \( r \) is the interest rate, and interest is being compounded continuously. Of course, \( \beta \) satisfies the deterministic equation

\[ \beta_t = \beta_0 + r \int_0^t \beta_s ds. \]

Consider a portfolio with \( a_t \) shares of stock, and \( b_t \) shares of the bond. The value of the portfolio at time \( t \) is

\[ V_t = a_t S_t + b_t \beta_t. \]

If \( a_t > 0 \), this means you’ve bought stock; if \( a_t < 0 \), you’re short-selling stock (which means you’ll need to buy it later). Similarly, if \( b_t < 0 \) then you’re borrowing money from the bank (and will need to repay it later). We’ll assume that \( a_t \) and \( b_t \) are bounded, there is no cost to transactions in the market, and you aren’t taking money out of the portfolio or adding it in.

The last condition is called self-financing, and it implies that

\[ dV_t = a_t dS_t + b_t d\beta_t = a_t (cS_t dt + \sigma S_t dB_t) + b_t (r \beta_t dt). \]

Note that we are missing terms from the formula: we should be writing

\[ dV_t = a_t dS_t + b_t d\beta_t + (da_t S_t + db_t \beta_t + da_t dB_t + db_t d\beta_t). \]

Now, \( db_t d\beta_t = 0 \) since \( d\beta_t \sim dt \), and \( da_t dS_t = 0 \) because \( a_t \) is predictable and therefore independent of the \( dB_t \) part of the increment \( dS_t \). The self-financing condition implies that \( S_t da_t + \beta_t db_t = 0 \): at time \( t \), you’re reallocating the money you have.

Let’s think this through in discrete time. There, at time \( n \) you look at the value of the portfolio, reallocate your assets from \((a_n, b_n)\) into \((a_{n+1}, b_{n+1})\) (which satisfy \( a_n S_n + b_n \beta_n = a_{n+1} S_{n+1} + b_{n+1} \beta_{n+1} \)), and then a price change happens and gives you \( V_{n+1} \). So:

\[ V_{n+1} = a_{n+1} S_{n+1} + b_{n+1} \beta_{n+1}, \quad V_n = a_n S_n + b_n \beta_n = a_{n+1} S_n + b_{n+1} \beta_n, \]

\[ \delta V = a_{n+1} \delta S + b_{n+1} \delta \beta. \]

Now consider an option, such as a European call, which at time \( T \) will be paying \((S_T - K)_+ = \max(S_T - K, 0)\). Let’s try to find a replicating portfolio for this option: that is, we want to find functions \( a_t, b_t \) that satisfy the conditions of a self-financing portfolio, and we need the final value to be equal to \((S_T - K)_+\).

Suppose that the value of this replicating portfolio has a particular form:

\[ V_t = a_t S_t + b_t \beta_t = u(T-t, S_t), \quad u \text{ a smooth deterministic function of two variables}. \]

That is, we assume that in order to price the call option we only need to know the remaining time and the current stock price. (This is reasonable in light of the independent increments property of Brownian motion, and it was true in discrete time.) The final value gives us an initial condition on \( u \):

\[ u(0, S_T) = (S_T - K)_+. \]

We’ll now derive a partial differential equation that \( u \) satisfies. The point is that we had two degrees of freedom: \( a_t \) and \( b_t \). We also have two equations: the self-financing
condition and the constraint on the form of $u$. This is enough to give us a differential equation on $u$. Let’s see this in detail.

Apply Itô lemma to $V = u(T - t, S_t)$:

$$dV_t = -\partial_t u(T - t, S_t) dt + \partial_S u(T - t, S_t) dS_t + \frac{1}{2} \partial_{SS} u(T - t, S_t) (dS_t)^2$$

$$= -\partial_t u(T - t, S_t) dt + \partial_S u(T - t, S_t) (c_S dt + \sigma S_t dB_t) + \frac{1}{2} \partial_{SS} u(T - t, S_t) (\sigma^2 S_t^2 dt)$$

$$= \left( -\partial_t u(T - t, S_t) + c_S \partial_S u(T - t, S_t) + \frac{1}{2} \sigma^2 S_t^2 \partial_{SS} u(T - t, S_t) \right) dt$$

$$+ (\sigma S_t \partial_S u(T - t, S_t)) dB_t.$$ 

This expresses the ordinary and the stochastic part of $dV$ in terms of $u$. We also had a way of expressing them in terms of $a$ and $b$:

\[
\begin{align*}
-\partial_t u(T - t, S_t) + c_S \partial_S u(T - t, S_t) + \frac{1}{2} \sigma^2 S_t^2 \partial_{SS} u(T - t, S_t) &= ca_t S_t + rb_t \beta_t, \\
\sigma S_t \partial_S u(T - t, S_t) &= \sigma a_t S_t,
\end{align*}
\]

This looks like a system of two equations in three unknowns, but remember that we have two possibilities.

Let’s solve: from the $dB$ term, $a_t = \partial_S u(T - t, S_t)$. Substituting this into the $dt$ term,

$$-\partial_t u(T - t, S_t) + \alpha a_t S_t + \frac{1}{2} \sigma^2 S_t^2 \partial_{SS} u(T - t, S_t) = \alpha a_t S_t + rb_t \beta_t = r(u(T - t, S_t) - a_t S_t).$$

Putting this together, we obtain the partial differential equation for $u$: for all $t > 0$ and $s > 0$,

$$\partial_t u(t, s) = \frac{1}{2} \sigma^2 s^2 \partial_{ss} u(t, s) + rs \partial_s u(t, s) - ru(t, s), \quad u(0, s) = (s - K)^+$$

Note that in continuous time, we really do have this for all $s > 0$, because the stock price can take any value (in discrete time, we needed to know there would only be two possibilities).

We haven’t used the condition on the final contract value yet: the PDE alone (without any initial conditions) is known as the Black-Scholes PDE, and it has to be satisfied by the value of any self-financing portfolio or contract in the Black-Scholes market. The initial condition (i.e. value at time $T$) is what determines the particular solution or contract.

In the case of $u(0, s) = (s - K)^+$, the combination of PDE and initial conditions has an explicit solution:

$$u(t, s) = s \Phi \left( \frac{\ln(s/K) + (r + \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \right) - Ke^{-rt} \Phi \left( \frac{\ln(s/K) + (r + \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} - \sigma \sqrt{t} \right),$$

where $\Phi$ is the normal CDF. Make sure you can check that this function $u(t, s)$ satisfies the PDE! We’ll see a way to derive this answer when we think about that calculation from a martingale point of view.

The replicating portfolio is given by

$$a_t = \partial_S u(T - t, S_t), \quad b_t = \frac{u(T - t, S_t) - a_t S_t}{\beta_t}.$$ 

So by observing the current stock price, you can figure out exactly how many shares of stock you’re meant to have in order to hedge the contract.

If we hadn’t managed to solve the final PDE explicitly, we could still approximate the solution numerically on the computer, by picking a grid of time increments and prices.
This is what you end up doing if you can’t get a neat answer for a particular contract. It’s nicer to do this for a non-random PDE because it’s easier to figure out whether it has converged.

4. A change-of-measure approach to Black-Scholes

In the previous section we found a replicating portfolio for a European call based on assuming that it had a particular form. We then used the SDEs satisfied by \( S_t \) and \( \beta_t \) to write down a non-random partial differential equation satisfied by the value \( u_t \), and we pulled a solution of that PDE out of a hat.

As we saw in the first lecture, there’s a different approach to arbitrage-free pricing in the market, and it’s based on the following idea:

1. Start by normalizing all prices so that the price of bonds is constant, i.e. a martingale.

2. Change measures to make the discounted price of stocks into a martingale. If we look at the answer we got by our first approach, we see that the mean rate of return \( c \) didn’t come into it at all: option price does not care about the mean rate of return, only about volatility. So an intelligent guess would be that we can make the discounted price of stocks look like \( \exp(-\sigma^2 t/2 + \sigma \tilde{B}t) \), the geometric Brownian motion that’s a martingale.

3. In the world where bonds and stocks are martingales, our financial derivative should be a martingale as well: then any portfolio will be a martingale, so its expected value will be the same at all times, so if there’s a chance of making a profit, there must also be a chance of losing. That’s exactly what “no arbitrage” is about.

So suppose that, after changing measure, we were able to write

\[
e^{-rt} S_t = e^{-\sigma^2 t/2 + \sigma \tilde{B}_t}
\]

for some Brownian motion \( \tilde{B}_t \). To make the discounted European call into a martingale, we will define

\[
e^{-rt} V_t = \mathbb{E} \left( e^{-rT} (S_T - K)_+ | \mathcal{F}_t \right),
\]

where \( \mathcal{F}_t = \sigma(S_u, u \leq t) \) gives the information available from the stock prices up to time \( t \). (As we saw by the tower law, conditioning a random variable on a filtration defines a martingale.)

As usual, decompose the stock price \( S_T \) into a part that’s \( \mathcal{F}_t \)-measurable and a part that’s independent of the filtration:

\[
e^{-rT} S_T = e^{-rt} S_t e^{-\sigma^2/2(T-t) + \sigma (\tilde{B}_T - \tilde{B}_t)}, \quad S_T = S_t e^{(r-\sigma^2/2)(T-t) + \sigma (\tilde{B}_T - \tilde{B}_t)}
\]

Then

\[
V_t = e^{-r(T-t)} \mathbb{E} \left( \underbrace{S_t e^{-\sigma^2/2(T-t) + \sigma (\tilde{B}_T - \tilde{B}_t)}}_{S_T} - K)_+ | \mathcal{F}_t \right).
\]

The point here is, as usual, that \( S_t \) is \( \mathcal{F}_t \)-measurable, whereas \( \sigma(\tilde{B}_T - \tilde{B}_t) \) is a normal random variable with mean 0, variance \( \sigma^2(T-t) \), and independent of \( \mathcal{F}_t \).

Recall the rule that said “if you have something independent inside a conditional expectation, you may take the average over it first”. Here that means computing

\[
\mathbb{E}[(x \mathcal{N}(0,\sigma^2(T-t))) - K)_+] \quad \text{as a function of } x = S_t e^{r-\sigma^2/2(T-t)}
\]
This is an integral we’ve looked at on a homework: you change variables to get a standard normal, and the “positive part” translates into bounds of integration. In the end, you’ll be able to get an answer that looks like “standard normal CDF evaluated at some points that depend on $S_t$ and $K$”. Since this expression will be $\mathcal{F}_t$-measurable, it’s the value $V(t, S_t)$, i.e.

$$V(t, S_t) = e^{-r(T-t)}\mathbb{E}_{N(0,\sigma^2(T-t))} \left[ (S_t e^{(r-\sigma^2/2)(T-t)} e^{N(0,\sigma^2(T-t))} - K)_+ \right],$$

or as an integral (notice that we’re integrating over $u$ below, which means that we treat $S_t$ as a constant!)

$$V(t, X_t) = e^{-r(T-t)} \frac{1}{\sqrt{2\pi\sigma\sqrt{T-t}}} \int_{-\infty}^{\infty} \left( S_t e^{(r-\sigma^2/2)(T-t)} e^u - K \right) e^{-\frac{1}{2} \frac{(u-K)^2}{(T-t)}} du.$$

If you change variables to get a standard normal in there, and figure out the bounds of integration that the “positive part” requires, you’ll get the Black-Scholes answer (try it!)

Note that we haven’t justified a step in this process: we claimed we should be able to change measure to get the stock price to look like $e^{-\frac{1}{2}\sigma^2 t + \sigma \tilde{B}_t}$, but we haven’t justified this yet.

**Theorem 4.1** (Girsanov’s theorem). Let $B_t$ be a Brownian motion, defined on the probability space $(\Omega, \mathbb{P})$. Fix a time interval $[0, T]$. There exists a different measure $Q$ on $\Omega$ such that the process

$$\tilde{B}_t = B_t + qt, \quad 0 \leq t \leq T$$

is a Brownian motion with respect to $Q$. (Meaning that, under $Q$, $\tilde{B}_t$ is still a Gaussian process, the $Q$-mean of $\tilde{B}_t$ is zero, and the $Q$-covariance between $\tilde{B}_t$ and $\tilde{B}_s$ is the same as under $\mathbb{P}$.)

The process $\tilde{B}_t$ is (obviously) adapted to the filtration of the original Brownian motion. (We used this when computing the conditional expectation: the filtration $\mathcal{F}$ that we’re conditioning on is the filtration of the original Brownian motion, i.e. the information we actually have at time $s$.)

The change of measure is given by

$$\frac{dQ}{d\mathbb{P}}(\omega) = e^{-qB_T(\omega) - \frac{1}{2}q^2 T}, \quad Q(A) = \int_A e^{-qB_T(\omega) - \frac{1}{2}q^2 T} d\mathbb{P}(\omega).$$