1. (1 point) Indicate all the points on the real line \((-\infty, \infty)\) where the following function is discontinuous.

\[
    f(x) = \begin{cases} 
        e^x & x \leq 0 \\
        \sin(x) & 0 < x < \frac{\pi}{4} \\
        \cos(x) & x \geq \frac{\pi}{4} 
    \end{cases}
\]

*Solution.* Recall that exponential, polynomial and trigonometric functions are always continuous. Therefore we only need to check continuity at the "bad points"; i.e. \(x = 0\) and \(x = \frac{\pi}{4}\).

i) At \(x = 0\)

\[
    \lim_{x \to 0^+} = \sin(0) = 0 \\
    \lim_{x \to 0^-} = e^0 = 1 \\
    f(0) = e^0 = 1
\]

Since \(\lim_{x \to 0^+} \neq \lim_{x \to 0^-}\) the function is discontinuous at \(x = 0\).

ii) At \(x = \frac{\pi}{4}\)

\[
    \lim_{x \to \frac{\pi}{4}^-} = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\
    \lim_{x \to \frac{\pi}{4}^+} = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\
    f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}
\]

Since all the values agree the function is continuous at \(x = \frac{\pi}{4}\).

2. (1 point) Find the derivative of the function \(f(x) = (x + 1)^4(x - 1)^7\) at the point \(x = 0\).
Solution. Using product rule:
\[
\frac{d}{dx} (x + 1)^4(x - 1)^7 = 4(x + 1)^3 \cdot (x - 1)^7 + (x + 1)^4 \cdot 7(x - 1)^6 \\
= 4(x + 1)^3(x - 1)^7 + 7(x + 1)^4(x - 1)^6
\]

Plug-in \(x = 0\)
\[
f'(0) = 4(0 + 1)^3(0 - 1)^7 + 7(0 + 1)^4(0 - 1)^6 \\
= 4(1)^3(-1)^7 + 7(1)^4(-1)^6 \\
= -4 + 7 \\
= 3
\]

\(\square\)

3. (1 point) Find the area of the region between the graph of the function \(f(x) = (1 - \sqrt{x})^2\) and the \(x\)-axis from \(x = 0\) to \(x = 1\).

Solution. We compute
\[
\int_0^1 (1 - \sqrt{x})^2 \, dx
\]
\[
\int_0^1 (1 - \sqrt{x})^2 \, dx = \int_0^1 (1 - 2\sqrt{x} + x) \, dx \\
= \int_0^1 x - \frac{4x^{3/2}}{3} + \frac{x^2}{2} \\
= \left[ x - \frac{4}{3}x^{3/2} + \frac{x^2}{2} \right]_0^1 \\
= \left( 1 - \frac{4}{3} + \frac{1}{2} \right) - (0) \\
= \frac{1}{6}
\]

\(\square\)

4. (1 point) Find an absolute maximum value of the function \(f(x) = 2 \ln x + 3x - x^2\) on the interval \((0, \infty)\).

Solution. Observe that \(f'(x) = \frac{2}{x} + 3 - 2x\). We find the critical points with the standard method.
\[
0 = \frac{2}{x} + 3 - 2x \\
\iff 0 = 2 + 3x - 2x^2 \\
\iff 0 = (-2x - 1)(x - 2) \\
\iff x = -\frac{1}{2} \text{ or } x = 2
\]

We observe that \(x = -1/2\) is outside the domain, therefore the only critical point is \(x = 2\). Furthermore, note that the concavity function \(f''(x) = -\frac{2}{x^2} - 2\) is always negative, therefore \(x = 2\) is the absolute maximum with \(f(2) = 2 \ln(2) + 2\).  

\(\square\)
5. (1 point) Compute

\[ \frac{d}{dx} \left( \sqrt{\frac{2x + 4}{x - 1}} \right) \]

**Solution.** The fastest way to compute the derivative is using logarithmic differentiation. Let

\[ y = \sqrt{\frac{2x + 4}{x - 1}} \]

then

\[ \ln(y) = \ln \left( \sqrt{\frac{2x + 4}{x - 1}} \right) = \frac{1}{2} \ln \left( \frac{2x + 4}{x - 1} \right) = \frac{1}{2} (\ln(2x + 4) - \ln(x - 1)) \]

We differentiate and see that

\[ \frac{y'}{y} = \frac{1}{2} \left( \frac{2}{2x + 4} - \frac{1}{x - 1} \right) \]

\[ \square \]

6. (1 point) Find

\[ \int_0^1 \frac{2}{1 + x^2} \, dx \]

**Solution.**

\[ \int_0^1 \frac{2}{1 + x^2} \, dx = 2 \left[ \tan^{-1}(x) \right]_0^1 = 2(\tan^{-1}(1) - \tan^{-1}(0)) = 2\left( \frac{\pi}{4} - 0 \right) = \frac{\pi}{2} \]

\[ \square \]

7. (1 point) Find the horizontal and vertical asymptotes of the function

\[ f(x) = \frac{1 + 4x^2}{1 - 4x^2} \]

**Solution.** We inspect the asymptotes separately.
i) Vertical asymptotes.
To find the vertical asymptotes we find the points at which the function is undefined, in our case that is when we divide by 0; i.e.

\[ 1 - 4x^2 = 0 \]
\[ \iff x^2 = \frac{1}{4} \]
\[ \iff x = \pm \frac{1}{2} \]

ii) Horizontal asymptotes.
To find the horizontal asymptotes we find the limit as \( x \) goes to infinity. We use L’Hopital rule, as it is the quickest way to proceed in general.

\[ \lim_{x \to \infty} \frac{1 + 4x^2}{1 - 4x^2} = \lim_{x \to \infty} \frac{8x}{-8x} = -1 \]

9. (1 point) Calculate the following limit

\[ \lim_{n \to \infty} \left(1 + \sin \left( \frac{1}{n} \right) \right)^{3n} \]
Solution. Let

\[ y = \lim_{n \to \infty} \left( 1 + \sin \left( \frac{1}{n} \right) \right)^{3n} \]

then

\[ \ln(y) = \ln \left( \lim_{n \to \infty} \left( 1 + \sin \left( \frac{1}{n} \right) \right)^{3n} \right) \]
\[ \ln(y) = \lim_{n \to \infty} \ln \left( 1 + \sin \left( \frac{1}{n} \right) \right)^{3n} \]
\[ \ln(y) = \lim_{n \to \infty} \left( 3n \ln \left( 1 + \sin \left( \frac{1}{n} \right) \right) \right) \]
\[ \ln(y) = 3 \lim_{n \to \infty} \left( n \ln \left( 1 + \sin \left( \frac{1}{n} \right) \right) \right) \]
\[ \ln(y) = 3 \lim_{n \to \infty} \left( \frac{\ln \left( 1 + \sin \left( \frac{1}{n} \right) \right)}{\frac{1}{n}} \right) \]

Observe that applying the limit directly leads to an indeterminate form \(0/0\). Then we use L’Hopital rule

\[ \ln(y) = 3 \lim_{n \to \infty} \left( \frac{\cos \left( \frac{1}{n} \right) \cdot \frac{-1}{n^2}}{1 + \sin \left( \frac{1}{n} \right) \cdot \frac{-1}{n^2}} \right) \]
\[ \ln(y) = 3 \lim_{n \to \infty} \left( \frac{\cos \left( \frac{1}{n} \right) \cdot \frac{-1}{n^2}}{1 + \sin \left( \frac{1}{n} \right) \cdot \frac{-1}{n^2}} \right) \]
\[ \ln(y) = 3 \lim_{n \to \infty} \left( \frac{\cos \left( \frac{1}{n} \right) \cdot \frac{-1}{n^2}}{1 + \sin \left( \frac{1}{n} \right) \cdot \frac{-1}{n^2}} \right) \]

As \( n \) goes to infinity \( \frac{1}{n} \) goes to 0 and we have

\[ \ln(y) = \lim_{n \to \infty} \left( \frac{\cos \left( \frac{0}{0} \right)}{1 + \sin \left( \frac{0}{0} \right)} \right) \]
\[ \ln(y) = \lim_{n \to \infty} \left( \frac{0}{1 + 0} \right) \]
\[ \ln(y) = \lim_{n \to \infty} \left( 0 \right) \]
\[ \ln(y) = \lim_{n \to \infty} \left( 1 \right) \]
\[ \ln(y) = 3 \]

take back the \( \ln \) and we have

\[ y = \exp^3 \]

\[ \square \]
10. (1 point) Let
\[ f(x) = x^{x^2 + 2}. \]
Compute \( f'(x) \).

\textit{Solution.} This is a standard logarithmic differentiation problem. Take the logarithmic function on both side and see that
\[
\frac{d}{dx}(\ln(f(x))) = \frac{d}{dx}((x^2 + 2) \ln(x))
\]
Then, using the product rule on the right
\[
\frac{f'(x)}{f(x)} = (2x) \ln(x) + (x^2 + 2) \left( \frac{1}{x} \right)
\]
Simplify if you wish.

11. (1 point) Let
\[ f(x) = \sqrt{\frac{e^x + 1}{2}}. \]
(a) (1 point) Linearize the function at \( x = 0 \).

\textit{Solution.} Compute \( f(0) \) and \( f'(0) \). You should get
\[
f(0) = 1 \quad \text{and} \quad f'(0) = \frac{1}{4}
\]
Therefore the linear approximation is
\[
L(x) = 1 + \frac{1}{4}x
\]

(b) (1 point) Estimate \( f(.1) \).

\textit{Solution.} Use the above formula, then
\[
f(.1) \approx L(.1) = 1 + \frac{1}{4}(.1)
\]

12. (1 point) Let \( f(x) = -x^3 + x^2 + x + 3 \).
(a) (1 point) Determine the intervals on which the function is increasing and decreasing.
\textit{Solution.} Observe that

\[ f'(x) = -3x^2 + 2x + 1 = -(3x + 1)(x - 1) \]

The roots of the derivative are \( x = -\frac{1}{3} \) and \( x = 1 \), therefore they split the domain into

\((-\infty, -\frac{1}{3}), (-\frac{1}{3}, 1)\) and \((1, \infty)\)

Choose any point in the domain to inspect the derivative (I chose \( x = -1, 0, 2 \)).

\[ f'(-1) = -4 \text{ therefore } f(x) \text{ is decreasing on the interval } (-\infty, -\frac{1}{3}) \]
\[ f'(0) = 1 \text{ therefore } f(x) \text{ is increasing on the interval } (-\frac{1}{3}, 1) \]
\[ f'(2) = -7 \text{ therefore } f(x) \text{ is decreasing on the interval } (1, \infty) \]

\(\square\)

(b) (1 point) Determine the intervals at which the graph is concave up and concave down.

\textit{Solution.} Observe that

\[ f''(x) = -6x + 2 \]

There is only one root of the function at \( x = \frac{1}{3} \). Therefore the concavity of the function splits the domain into the intervals \((-\infty, \frac{1}{3})\) and \((\frac{1}{3}, \infty)\). Pick points in the respective intervals to determine the sign of concavity. e.g.

\[ f''(0) = 2 \text{ therefore } f(x) \text{ is concave up on the interval } (-\infty, \frac{1}{3}) \]
\[ f''(1) = -4 \text{ therefore } f(x) \text{ is concave down on the interval } (\frac{1}{3}, \infty) \]

\(\square\)

(c) (1 point) Determine the points at which the graph achieves local maximum and minimum values.

\textit{Solution.} With the information from part (a) and part (b) it should be easy to see that \( x = -\frac{1}{3} \) is a local minimum and \( x = 1 \) is a local maximum.

\(\square\)

(d) (1 point) Sketch the graph of \( f(x) \) on the interval \([-1, 2]\).

\textit{Solution.} Sketch...

13. (1 point) Calculate

\[ \int_{0}^{\pi/2} (\cos^2 x - 2 \cos x) \sin x \, dx. \]
Solution. Use the substitution \( u = \cos(x) \) and observe that
\[
\int_0^{\pi/2} (\cos^2 x - 2\cos x) \sin x \, dx = -\int_1^0 (u^2 - 2u) \, du
\]
Then,
\[
-\int_1^0 (u^2 - 2u) \, du = -\int_1^0 \frac{u^3}{3} - u^2 = -\left( \frac{1}{3} - \frac{1}{3} \right) = -\frac{2}{3}
\]

14. (1 point) Find the indefinite integral
\[
\int \frac{2x + e^{-\sqrt{x+1}}}{\sqrt{x+1}} \, dx.
\]
Solution. Use the substitution \( u = \sqrt{x+1} \), then we have
\[
\int \frac{2x + e^{-\sqrt{x+1}}}{\sqrt{x+1}} \, dx = \int \left( 2(u^2 - 1) + e^{-u} \right) \cdot 2 \, du
\]
\[
= \int 4u^2 - 4 + 2e^{-u} \, du
\]
\[
= \frac{4u^3}{3} - 4u - 2e^{-u}
\]
You might take the u-substitution back if you wish.

15. (1 point) Let
\[
g(x) = \int_1^x \ln(t^2 - 1) \, dt.
\]
(a) (1 point) Find the first and second derivatives of \( g \).

Solution. We make use of the fundamental theorem of calculus and see that
\[
g'(x) = \ln(x^2 - 1)
\]
and
\[
g''(x) = \frac{2x}{x^2 - 1}
\]

(b) (1 point) Find the \( x \) value of the maximum and minimum points of \( g \) on the interval \([1, 10] \).
Solution. This is a standard "set the derivative equal to 0" problem. We have
\[
g'(x) = 0 \\
\ln(x^2 - 1) = 0 \\
x^2 - 1 = e^0 \\
x^2 = 2 \\
x = \pm\sqrt{2}
\]
Note that \(-\sqrt{2}\) is outside the domain, then, we only need to inspect the critical value \(x = +\sqrt{2}\). Use the second derivative and see that:
\[g''(\sqrt{2}) = 2\sqrt{2} > 0 \quad \text{therefore } \sqrt{2} \text{ is a minimum.}\]

16. (1 point) Calculate the following limit, if it exists. Otherwise, explain why it does not exist.
\[
\lim_{x \to 1} \frac{\ln(x)}{\cos(\frac{\pi}{2} x)}.
\]
Solution. Applying the limit directly leads to the indeterminate form \(0 \div 0\) therefore we may use L’Hopital.
\[
\lim_{x \to 1} \frac{\ln(x)}{\cos(\frac{\pi}{2} x)} = \lim_{x \to 1} \frac{\frac{1}{x}}{-\frac{\pi}{2} \sin(\frac{\pi}{2} x)}
\]
Apply the limit and see that
\[
\lim_{x \to 1} \frac{\frac{1}{x}}{-\frac{\pi}{2} \sin(\frac{\pi}{2} (1))} = \frac{1}{-\frac{\pi}{2}} = -\frac{2}{\pi}
\]

17. (1 point) The graph of the equation \(x^2 - 2xy - y^2 - x = 2\) is a hyperbola. Verify that the point \((-1, 2)\) is on this hyperbola, and find the equation of the tangent line to the graph at that point.

Solution. We will use the point slope formula
\[y - y_0 = m(x - x_0)\]
and since the point \((-1, 2)\) is given, it suffices to find \(m\), or equivalently, the derivative \(y'\). Observe that the given equation is equivalent to
\[(x - y)^2 - 2y^2 = 2\]
then
\[ \frac{d}{dx}((x-y)^2 - 2y^2) = \frac{d}{dx}(2) \]
\[ 2(x-y)(1-y') - 4yy' = 0 \]
Plug-in the given point i.e. \( x = -1 \) and \( y = 2 \) to get
\[ 2(-1 - 2)(1 - y') - 4(2)y' = 0 \]
\[ 2(-3)(1 - y') - 8y' = 0 \]
\[ -6(1 - y') - 8y' = 0 \]
\[ -6 + 6y' - 8y' = 0 \]
\[ -2y' = 6 \]
\[ y' = -3 \]
Therefore the equation of the tangent line is
\[ y - 2 = -3(x + 1) \]

18. (1 point) The width of a rectangle is growing at the rate of 2 inches per minute, and its height is getting smaller at the rate of 3 inches per minute. Find the rate of change of the length of the diagonal of the rectangle when the width is 20 inches and the height is 15.
(Note: when the width is 20 and the height is 15, the diagonal is 25 inches.)

Solution. Let \( w \) be the width variable and \( h \) be the height variable. Since we have a rectangle we have a Pythagorean identity along the diagonal; i.e.
\[ d^2 = w^2 + h^2. \]
Then, take the implicit derivative with respect to time
\[ \frac{d}{dt}(d^2) = \frac{d}{dt}(w^2 + h^2) \]
\[ 2d \cdot d' = 2w \cdot w' + 2h \cdot h' \]
Make the appropriate substitutions and see that
\[ 2(25)d' = 2(20)(2) + 2(15)(-3) \]
\[ 50d' = 80 - 90 \]
\[ 50d' = -10 \]
\[ d' = \frac{-10}{50} = -\frac{1}{5} \]