Fourier Series and Distribution Theory

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Abstract

Lecture notes taken during the Fall 2018 semester at the University of Minnesota. The following notes are relevant to the course Mathematical Modeling and Methods of Applied Mathematics as taught by Dr. Lai, Ru-Yu. These notes are not intended to be a formal manuscript rather, a reference. The author is responsible for any errors herein.

1 Fourier Series

1.1 Preliminary Results

Consider an interval \( I \in \mathbb{R} \), closed and finite. We say \( f \in C_I \) if \( f \) is continuous in \( I \). We define the norm of \( f \) to be

\[
\|f\|_{\infty} = \max_{x \in I} |f(x)|
\]

Definition 1.1 (Uniform convergence). We say that the sequence \( f_n \) converges uniformly to \( f \) in \( I \) if, for \( \varepsilon > 0 \), there exists \( N \) independent if \( x \) such that

\[
|f_n - f| < \varepsilon
\]
for all \( n > N \). This is equivalent to \( \| f_n - f \|_\infty \to 0 \) as \( n \to \infty \).

**Lemma 1.1.** If \( f_n \) converges uniformly to \( f \) and \( f_n \in C_I \) then, \( f \in C_I \).

**Proof.** Since \( f_n \to f \) uniformly then, for \( \epsilon > 0 \) there exist \( N \) such that
\[
|f_n - f| < \epsilon
\]
for all \( x \) whenever \( n > N \). Now, consider \( f_N \in C_I \). By definition of continuity there exist \( \delta \) such that
\[
|f_N(x) - f_N(y)| < \epsilon
\]
whenever \( |x - y| < \delta \). Then, observe that
\[
|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \leq 3\epsilon
\]
analogous to what we wanted. \( \square \)

**Lemma 1.2** (limit and integral interchange). If \( f_n \) converges uniformly to \( f \) over a compact interval \( I \), then
\[
\lim_{n \to \infty} \int_I f_n = \int_I f = \int_I \lim_{n \to \infty} f_n
\]

**Proof.** Note that
\[
\left| \int_I f_n - \int_I f \right| \leq \int_I |f_n - f| \leq \int_I \| f_n - f \|_\infty \leq \| f_n - f \|_\infty |I| < \epsilon.
\]

**Lemma 1.3.** If \( f_n \in C_I^1 \), and \( f_n \) converges to \( f \) uniformly, and moreover \( f'_n \) converges to \( g \), then \( f' \) converges to \( g \) and \( f \in C_I^1 \).

**Proof.** Since \( f_n \in C_I^1 \) then
\[
f_n(x) = f_n(x_0) + \int_{x_0}^x f'_n(t) \, dt.
\]
Then, taking the limit as \( n \) goes to infinity we get
\[
f(x) = f(x_0) + \int_{x_0}^x g(t) \, dt.
\]
By differentiation \( f'(x) = g(x) \). \( \square \)
1.2 Convergence of Fourier Series

Let $T = \mathbb{R}/2\pi n$, with $n \in \mathbb{N}$. As before $C(T)$ is to imply continuity on $T$.

**Definition 1.2** (Inner product on $C(T)$). We define the inner product of two functions $f, g$ in $C(T)$ as:

$$\langle f, g \rangle := \int_T f(x) g(x) \, dx.$$  

It should be clear that the definition satisfies the properties of an inner product, namely

1. $\langle f, f \rangle > 0$ whenever $f \neq 0$, and otherwise $\langle f, f \rangle = 0$.
2. $\langle f, g \rangle = \overline{\langle g, f \rangle}$.
3. $\langle \alpha f + g, h \rangle = \alpha \langle f, h \rangle + \langle g, h \rangle$

Then, the induced norm is given by:

$$|f|^2 = \int_T |f|^2 \, dx.$$  

Now, consider the functions

$$\psi_k := \frac{1}{\sqrt{2\pi}} e^{ikx}$$

for $k \in \mathbb{Z}$. Observe that these functions are in $C(T)$ and moreover they form an orthonormal basis.

**Definition 1.3** (Fourier coefficient). For $k \in \mathbb{Z}$, the $k$th Fourier coefficient is defined as

$$\hat{f}(k) := \langle f, \psi_k \rangle.$$  

**Definition 1.4** (Fourier series).

$$f \sim \sum_{k \in \mathbb{Z}} \langle f, \psi_k \rangle \psi_k = \sum_{k \in \mathbb{Z}} \hat{f}(k) \psi_k.$$  

**Example.** Consider the function

$$f(x) = \begin{cases} 
-1 & x < 0 \\
0 & x = 0 \\
1 & x > 0
\end{cases}.$$  

Its Fourier series is computed to be

$$f \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nx)}{n}.$$
Proposition 1.1. If \( f \in C^r(T) \), then \( \langle f^{(r)}, \psi_k \rangle = (ik)^r \langle f, \psi_k \rangle \). Equivalently \( \hat{f}^{(r)}(k) = (ik)^r \hat{f}(k) \).

Proof. By integration by parts.

Remarks

i. Derivatives of \( f \) are mapped to multiples of \( f \).

ii. \( \hat{f}(k) \sim O(|k|^{-r}) \).

iii. Smoothness of \( f \) corresponds to the decay property of \( \hat{f}(k) \) for large \( k \).

Definition 1.5 (Fourier partial sum).

\[
S_n(f) := \sum_{|k| \leq n} \hat{f}(k) \psi_k.
\]

Proposition 1.2. If \( f \in C(T) \), then \( \|S_n(f)\|_2 \leq \|f\|_2 \).

Proof. Note that

\[
\|f\|_2^2 = \int_T |f|^2 = \langle f, f \rangle = \langle (f - S_n(f)) + S_n(f), (f - S_n(f)) + S_n(f) \rangle = |f - S_n(f)|^2 + \langle f - S_n(f), S_n(f) \rangle + \langle S_n(f), f - S_n(f) \rangle + |S_n(f)|^2
\]

It suffices to prove that \( \langle f - S_n(f), S_n(f) \rangle \) and \( \langle S_n(f), f - S_n(f) \rangle \) are non-negative. In fact, these terms are equal to 0. Using the definition of \( S_n \) we have \( \langle S_n(f), f - S_n(f) \rangle = 0 \) and \( \langle S_n(f), f \rangle = |S_n(f)|^2 \). Thus, \( \langle S_n(f), f - S_n(f) \rangle = 0 \).

Lemma 1.4 (Best approximation).

\[
\|f - S_n(f)\|_2 \leq \left\| f - \sum_{k \in \mathbb{Z}} c_k \psi_k \right\|_2.
\]

Moreover, equality is attained if \( c_k = \hat{f}(k) \).

Proposition 1.3 (Bessel’s inequality). For \( f \in C(T) \)

\[
\sum_{k \in \mathbb{Z}} |\langle f, \psi_k \rangle|^2 \leq \|f\|_2^2.
\]

As a corollary we have that \( |\langle f, \psi_k \rangle|^2 \) goes to 0 as \( n \) goes to infinity.
Lemma 1.5 (Riemann-Lebesgue).
\[
\lim_{k \to \infty} \int_{T} f(x) \cos(kx) \, dx = \lim_{k \to \infty} \int_{T} f(x) \sin(kx) \, dx = 0.
\]

**Theorem 1.1.** If \( f \in C^1(T) \), then \( S_n(f) \) converges to \( f \) uniformly.

**Proof.** We prove the statements in two parts. First, we prove that indeed, \( S_n(f) \) converges to \( f \). Then, we prove uniformity by showing that \( \| S_n(f) - f \|_\infty \to 0 \) as \( n \to \infty \).

**Convergence.** Note that
\[
S_n(f) = \sum_{|k| \leq n} \langle f, \psi_k \rangle \psi_k
\]
\[
= \sum_{|k| \leq n} \left( \int_{T} f(y) \frac{\exp(-iky)}{\sqrt{2\pi}} \, dy \right) \frac{\exp(-iky)}{\sqrt{2\pi}}
\]
\[
= \int_{T} f(y) \left( \frac{1}{2\pi} \sum_{|k| \leq n} e^{ik(x-y)} \right) \, dy.
\]

We identify the term inside the latter parenthesis as the Dirichlet kernel \( D_n(x-y) \), where
\[
D_n(x) := \frac{1}{2\pi} \frac{\sin \left( (n + \frac{1}{2}) y \right)}{\sin \left( \frac{y}{2} \right)}.
\]

Then, the change of variables \( z = x-y \) leads to
\[
S_n(f) = \int_{T} f(x-z) D(z) \, dz
\]
\[
= (f * D_n)(x).
\]

Take for granted that
\[
\int_{T} D_n(y) \, dy = 1,
\]
therefore \( f \) can be written as
\[
f(x) = \int_{T} f(x) D_n(y) \, dy.
\]

Therefore
\[
|S_n(f) - f| = \left| \int_{T} (f(x-y) - f(x)) D_n(y) \, dy \right|
\]
\[
= \frac{1}{2\pi} \int_{T} \frac{f(x-y) - f(x)}{\sin(y/2)} \sin \left( \left( n + \frac{1}{2} \right) y \right) \, dy
\]
the term \( \frac{f(x-y)-f(x)}{\sin(y/2)} \) can be proven to be continuous whenever \( \sin(y/2) \neq 0 \). Then, by Riemann-Lebesgue lemma, the integral decays to 0 as \( n \to \infty \). Proving convergence to \( f \).

**Uniformity of the convergence.** We will use the convergence of \( S_n(f) \) to \( f \) and show that \( \|S_n(f) - S_m(f)\|_\infty \to 0 \) as \( n, m \to \infty \). Note that

\[
\|S_n(f) - S_m(f)\|_\infty = \left\| \sum_{n < |k| < m} \hat{f}(k) \psi_k \right\|_\infty,
\]

then,

\[
\left\| \sum_{n < |k| < m} \hat{f}(k) \psi_k \right\|_\infty \leq \frac{1}{\sqrt{2\pi}} \sum_{n < |k| < m} \left\| \hat{f}(k) \right\|_\infty \\
\leq \sum_{n < |k| < m} \frac{1}{|k|} \left\| \hat{f}'(k) \right\|_\infty \\
\leq \sum_{n < |k| < m} \frac{1}{|k|} \left\| \hat{f}'(k) \right\|_\infty.
\]

Now, using the Cauchy-Schwartz inequality, the latter satisfies

\[
\sum_{n < |k| < m} \frac{1}{|k|} \left\| \hat{f}'(k) \right\|_\infty \leq \left( \sum_{n < |k| < m} \frac{1}{|k|^2} \right)^{1/2} \left( \sum_{n < |k| < m} \left\| \hat{f}'(k) \right\|_\infty^2 \right)^{1/2}
\]

Taking the limit as \( m \to \infty \) the first sum may be bounded by \( \sqrt{n}^{-1} \) and the latter, by \( \|f'\|_2 \). Thus,

\[
\|S_n(f) - f\|_\infty \leq \frac{\|f'\|_2}{\sqrt{n}}.
\]

Finally, in the limit as \( n \to \infty \), \( \|S_n(f) - f\|_\infty \to 0 \), as we wanted to show. \( \square \)

**Proposition 1.4.** If \( f \in L^2(T) \) and \( g_x(y) = \frac{f(x)-f(x-y)}{2\pi \sin(y/2)} \in L^2(T) \), then \( S_n(f) \) converges to \( f \) as \( n \to \infty \).

**Proof.** Analogous to that of Theorem 1.1. \( \square \)

**Proposition 1.5.** If \( f \in C^r(T) \), and \( r \geq 1 \) then,

\[
\lim_{n \to \infty} n^{r-1/2} \|S_n(f) - f\|_\infty = 0.
\]

**Proof.** Note that

\[
|S_n(f) - S_m(f)| \leq \sum_{n \leq k \leq m} \left| \hat{f}(k) \right| |e^{ikx}|,
\]

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and recall that $|\hat{f}(k)| \leq |\hat{f}(r)|/|k|^r$. Then,

$$
\sum_{n \leq k \leq m} |\hat{f}(k)| e^{ikx} \leq \sum_{n \leq k \leq m} \frac{|\hat{f}(r)(k)|}{|k|^r}.
$$

Now we use the Cauchy-Schwartz inequality, we get

$$
\sum_{n \leq k \leq m} \frac{|\hat{f}(r)(k)|}{|k|^r} \leq \left( \sum_{n \leq k \leq m} \frac{1}{|k|^{2r}} \right)^{1/2} \left( \sum_{n \leq k \leq m} |\hat{f}(r)(k)|^2 \right)^{1/2} \leq \frac{n^{1-2r}}{2r-1} \left( \sum_{n \leq k \leq m} |\hat{f}(r)(k)|^2 \right)^{1/2}.
$$

It follows that

$$
n^{2r-1} |S_n(f) - S_m(f)| \leq \frac{1}{2r-1} \left( \sum_{n \leq k \leq m} |\hat{f}(r)(k)|^2 \right)^{1/2},
$$

and therefore, as $m \to \infty$,

$$
n^{2r-1} \|S_n(f) - f\|_\infty \leq \frac{1}{2r-1} \left( \sum_{n \leq k} |\hat{f}(r)(k)|^2 \right)^{1/2}.
$$

Now, in the limit $n \to \infty$, the right hand side is the tail of the Fourier partial sum, which is convergent, thus its limit is 0 as we wanted.

Remark Smoothness of $f$ translates to faster convergence of the Fourier Series as

$$
\|S_n(f) - f\| < \frac{1}{n^{r-1/2}}.
$$

Theorem 1.2 ($L^2$ convergence). If $f \in C^1(T)$ then, $\|S_n(f) - f\|_2 \to 0$ as $n \to \infty$.

Before we write the proof we need to introduce the idea of mollifiers.

Definition 1.6 (Mollifier). We say $\varphi$ is a mollifier if $\varphi \in C^\infty$, and $\int \varphi = 1$.

e.g.

$$
\varphi(x) = \begin{cases} 
\exp \left( \frac{1}{|x|^2-1} \right) & |x| \leq 1 \\
0 & |x| > 1
\end{cases}.
$$

Now, let $\varphi(x)$ be an arbitrary mollifier, and let

$$
\varphi_\varepsilon(x) = \frac{\varphi(x)}{\varepsilon}.
$$
Lemma 1.6. Let \( f(x) = f \ast \varphi_\varepsilon(x) \), where \( \ast \) is the convolution operator, then \( f \in C^\infty \) for any choice of \( \varepsilon \).

Sketch of the proof. For fixed \( \varepsilon \):

i) \( f \in C^1 \) using the definition of the derivative.

ii) Note that \( f'(x) = f \ast \varphi'_\varepsilon(x) \).

iii) For higher derivatives the argument is similar to that in i)

Lemma 1.7. If \( f \in C(T) \) then, for all \( \varepsilon > 0 \) there exist \( g \in C^\infty(T) \) such that
\[
\|f - g\|_\infty < \varepsilon.
\]

Proof. The intuition is to compute \( f \ast \varphi \) for an appropriate mollifier \( \varphi \).

Lemma 1.8. If \( f \in C(T) \) then, we can find \( g \in C^\infty(T) \) such that
\[
\|f - g\|_2 < \varepsilon
\]
for arbitrary \( \varepsilon > 0 \).

Proof. By Lemma 1.7, we choose \( g \) such that
\[
\|f - g\|_\infty < \varepsilon
\]
then,
\[
\|f - g\|_2^2 = \int_T |f - g|^2 \leq \|f - g\|_\infty^2 \leq 2\pi \varepsilon^2.
\]

Theorem 1.3. If \( f \in C(T) \) then,
\[
\|S_n(f) - f\|_2 \to 0
\]
as \( n \to \infty \).

Proof. Since \( f \in C(T) \) we can find \( g \in C^\infty(T) \) such that
\[
\|f - g\|_\infty < \varepsilon.
\]
Moreover, since \( g \in C^\infty(T) \), we have that its Fourier partial sum converges uniformly; namely,
\[
\|S_n(g) - g\|_\infty \to 0
\]
as \( n \to \infty \). Now, note the following,

\[
\|S_n(f) - f\|_2 \leq \|S_n(f) - S_n(g)\|_2 + \|S_n(g) - g\|_2 + \|g - f\|_2 \leq 3\varepsilon.
\]

\[\square\]

1.3 The Heat Equation in \( \mathbb{R}/2\pi\mathbb{Z} \)

Consider

\[
\begin{aligned}
\frac{\partial}{\partial t}[u(x, t)] &= \frac{\partial^2}{\partial x^2}[u(x, t)] & t > 0, x \in \mathbb{R}/2\pi\mathbb{Z} \\
u(x, 0) &= f(x) & f(x) \in C(T)
\end{aligned}
\]

(1)

Without being rigorous for the moment, we assume that \( u(x, t) \) has the form

\[
u(x, t) = \sum_{k \in \mathbb{Z}} u_k(t)\psi_k(x),\]

where \( \psi_k(x) = \exp(ikx) \). Substitution into (1) leads to

\[
\sum_{k \in \mathbb{Z}} \frac{\partial}{\partial t}[u_k(t)]\psi_k(x) = \sum_{k \in \mathbb{Z}} -k^2 u_k(t)\psi_k(x).
\]

By taking the inner product against \( \psi_k(x) \) we see that, for each \( k \), \( u_k(t) \) satisfies

\[
\frac{\partial}{\partial t}[u_k(t)] = -k^2 u_k(t).
\]

Using elementary methods the solution is of the form

\[
u_k(t) = C_k(x) \exp(-k^2 t).
\]

Then, using the initial condition in (1) we have

\[
u(x, 0) = \sum_{k \in \mathbb{Z}} u_k(0)\psi_k(x) = f(x),
\]

therefore

\[
\sum_{k \in \mathbb{Z}} C_k(x)\psi_k(x) = f(x).
\]

Now, if we take the inner product against \( \psi_k(x) \) we see that

\[
C_k(x) = \langle f(x), \psi_k(x) \rangle.
\]
It follows that
\[ u(x,t) = \sum_{k \in \mathbb{Z}} \langle f, \psi_k \rangle \exp(-k^2t) \psi_k(x). \]

(2)

We would like to formally show that (2)

i) Converges to (1)

ii) Satisfies the initial condition

First we introduce a helpful result.

**Proposition 1.6** (Weierstrass M-test). Given a sequence \( f_n(x) \) such that, for any \( n \), \( \|f_n(x)\|_\infty \leq M_n \) for some \( M_n \) then, if \( \sum_n M_n \) converges, then \( \sum_n f_n(x) \) converges uniformly.

**Remark.** For \( t > 0 \), the solution \( u(x,t) \) as in (2) is in \( C^\infty(T) \). Moreover,
\[
\frac{\partial^m}{\partial t^m} \frac{\partial^\ell}{\partial x^\ell} [u(x,t)] = \sum_{k \in \mathbb{Z}} (-k^2)^m (ik)^\ell e^{-k^2t} \psi_k(x).
\]

We proceed to show that (2) satisfies
\[
\frac{\partial}{\partial t} [u(x,t)] = \frac{\partial^2}{\partial x^2} [u(x,t)]
\]

**Proof.** From the remark above let \( m = 1 \) and \( \ell = 0 \), then
\[
\frac{\partial}{\partial t} [u(x,t)] = \sum_{k \in \mathbb{Z}} (-k^2) \langle f, \psi_k \rangle \exp(-k^2t) \psi_k(x),
\]
equivalently,
\[
\frac{\partial}{\partial t} [u(x,t)] = \lim_{n \to \infty} \sum_{|k|<n} (-k^2) \langle f, \psi_k \rangle \exp(-k^2t) \psi_k(x).
\]

Now, for \( t > 0 \) let the sequence \( v_n \) be defined as
\[
v_n := \sum_{|k|<n} (-k^2) \langle f, \psi_k \rangle \exp(-k^2t) \psi_k(x).
\]

Then, by Proposition 1.6 \( v_n \) converges uniformly. Furthermore, eote that, from (2) we have
\[
\left| \langle f, \psi_k \rangle e^{-k^2t} \psi_k(x) \right| \leq \frac{C_0}{\sqrt{2\pi}} e^{-k^2t},
\]
therefore, by Proposition 1.6 again, (2) also converges uniformly. Altogether we have
i.) Expression (2) converges uniformly to $u(x,t)$, and

ii.) The partial derivative of $(\partial_t)$ of (2), converges uniformly to a function.

Invoking Lemma 1.3 we have that

$$\frac{\partial}{\partial t}[u(x,t)] = \sum_{k \in \mathbb{Z}} (-k^2) \langle f, \psi_k \rangle \exp(-k^2t) \psi_k(x).$$

With a similar argument we can find the partial derivative $\partial_x[u]$. Moreover, we find that indeed, the proposed solution to (1) satisfies

$$\frac{\partial}{\partial t}[u(x,t)] = \frac{\partial^2}{\partial x^2}[u(x,t)]$$

as we wanted. \hfill \Box

It remains to show that (2) satisfies the initial condition. Let

$$H_t(f) := \sum_{k \in \mathbb{Z}} \langle f, \psi_k \rangle \exp(-k^2t) \psi_k(x).$$

**Theorem 1.4.** The solution (2) to the heat equation, satisfies initial conditions $u(x,0) = f(x) \in C(T)$. Equivalently,

$$\lim_{t \to 0} |H_t(f) - f| = 0.$$

**Sketch of the proof.** We estimate $f$ with an appropriate function $g$ and find $\varepsilon$ bounds for the terms in the expression

$$|H_t(f) - f| \leq \underbrace{\|H_t(f) - H_t(g)\|}_0 + \underbrace{\|H_t(g) - g\|}_0 + \underbrace{\|g - f\|}_0 +$$

It can be shown that if $g \in C^2(T)$ the term (II) goes to 0. Then, it can also be shown that (I) satisfies

$$\|H_t(f) - H_t(g)\|_\infty \leq \|f - g\|_\infty.$$ 

Thus, it suffices to choose $g \in C^2(T)$ such that

$$\|f - g\|_\infty \leq \varepsilon.$$

\hfill \Box
1.4 Laplace equation

We consider the Laplace equation

\[ \begin{cases} \Delta u(x) = 0 & \text{for } |x| < 1, \\ u(x) = f(x) & \text{for } |x| = 1. \end{cases} \tag{3} \]

In polar coordinates

\[ \begin{cases} \frac{\partial^2}{\partial r^2} u(r, \theta) + \frac{1}{r} \frac{\partial}{\partial r} u(r, \theta) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} u(r, \theta) = 0 & \text{for } r < 1, \\ u(1, \theta) = f(\theta) \in C^2(T). \end{cases} \tag{4} \]

As before, we expect the solution to be of the form:

\[ u(r, \theta) = \sum_{k \in \mathbb{Z}} u_k(r) \psi_k(\theta). \tag{5} \]

Direct substitution of (5) into (4) leads to the problem

\[ \frac{\partial^2}{\partial r^2} u_k(r) + \frac{1}{r} \frac{\partial}{\partial r} u_k(r) - \frac{k^2}{r^2} u_k(r) = 0; \]

this equation is known as Euler’s equation. By theory of ODEs, we find that the solution to the previous equation is

\[ \begin{align*} u_k &= A_k(\theta) r^{|k|} + B_k(\theta) r^{-|k|}, & \text{when } k \neq 0, \\ u_k &= A_0 + B_0 \ln(r), & \text{when } k = 0. \end{align*} \]

It follows that \( u(r, \theta) \) must be of the form

\[ u(r, \theta) = (A_0 + B_0 \ln(r)) \psi_0(\theta) + \sum_{k \in \mathbb{Z}} (A_k(\theta) r^{|k|} + B_k(\theta) r^{-|k|}) \psi_k(\theta). \]

We assume that \( u \) is bounded for \( r = 0 \) thus, \( B_k = 0 \) for all \( k \). Then, \( u(k, \theta) \) is of the form

\[ u(r, \theta) = A_0 \psi_0(\theta) + \sum_{k \in \mathbb{Z}} (A_k(\theta) r^{|k|}) \psi_k(\theta). \]

Moreover, in order to satisfy the boundary condition \( u = f(\theta) \) as \( r \to 1 \) it must be the case that

\[ f(\theta) = \sum_{k \in \mathbb{Z}} A_k(\theta) \psi_k(\theta). \]

Now, we take the inner product against \( \psi_j \) for a fixed \( j \). We see that

\[ \langle f, \psi_j \rangle = A_j(\theta). \]
Altogether, the solution to the Laplace equation in polar coordinates (4) is:

\[ u(r, \theta) = \sum_{k \in \mathbb{Z}} \langle f, \psi_k \rangle r^{|k|} \psi_k(\theta). \]  

(6)

Note that \( u(r, \ell) \) as above converges as

\[
|u(r, \theta)| \leq \sum_{k \in \mathbb{Z}} |\langle f, \psi_k \rangle| r^{|k|} |\psi_k(\theta)| \\
\leq \sum_{k \in \mathbb{Z}} \frac{1}{k^2} r^{|k|} K \\
\leq \infty.
\]

The Poisson kernel  We will write the solution (6) as a convolution. Note that

\[
u(r, \theta) = \sum_{k \in \mathbb{Z}} \langle f, \psi_k \rangle r^{|k|} \psi_k(\theta)
= \sum_{k \in \mathbb{Z}} \left( \frac{1}{2\pi} \int_{T} f(y) e^{-ik(y-\theta)} \, dy \right) \, r^{|k|} e^{ik\theta}
= \frac{1}{2\pi} \int_{T} \left( \sum_{k \in \mathbb{Z}} r^{|k|} e^{-ik(y-\theta)} f(y) \right) \, dy.
\]

We define the Poisson kernel \( P_r(\theta) \) as the sum in the above expression. Then,

\[
u(r, \theta) = \frac{1}{2\pi} \int_{T} P_r(\theta - y) \, f(y) \, dy
= \frac{1}{2\pi} (P_r * f)(\theta),
\]

Lemma 1.9. The Poisson kernel satisfies

\[ P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos(\theta) + r^2}. \]

Moreover,

\[
\frac{1}{2\pi} \int_{T} P_r(\theta) \, d\theta = 1.
\]

Proof. Note that the series definition of the Poisson kernel is geometric, using elementary formulas we can arrive at the desired result.

\[ \square \]

Theorem 1.5. The expression

\[ u(r, \theta) = \frac{1}{2\pi} (P_r * f)(\theta) \]

solves the Laplace equation (4).
**Theorem 1.6.** The expression

\[ u(r, \theta) = \frac{1}{2\pi} (P_r * f)(\theta) \]  

satisfies the boundary condition of (4).

**Proof.** We will show that (7) converges uniformly to \( f(\theta) \). Namely, we will show that

\[ \lim_{r \to 1} \| u(r, \theta) - f(\theta) \|_{\infty} = 0 \]

Note that

\[
|u(r, \theta) - f(\theta)| = \left| \sum_{k \in \mathbb{Z}} (f, \psi_k) \psi_k (r|k| - 1) \right|
\]

\[ \leq \sum_{k \in \mathbb{Z}} \frac{1}{|k|^2} (1 - r|k|) \]

\[ \leq \sum_{k < |N|} \frac{1}{|k|^2} (1 - r|k|) + \sum_{k \geq N} \frac{1}{|k|^2}. \]

Note that the expressions are independent of \( \theta \). Moreover, in the limit as \( N \to \infty \) the right sum converges to 0, while in the limit \( r \to 1 \) the left sum approaches 0 as well.  

## 2 Distribution theory

### 2.1 Definitions and notation

Let \( \Omega \) be an open set in \( \mathbb{R}^n \). Denote the partial derivative in the direction of \( x_j \) to be

\[ \partial_j f = \frac{\partial}{\partial x_j} \]

for any choice of \( j \in \{1, \ldots, n\} \).

For higher order derivatives we introduce a vector \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), then

\[ \partial^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f. \]

e.g. Let \( f(x, y) = y^3 \exp(x^2) + y^7 \), and \( \alpha = (1, 3) \). Then,

\[ \partial^\alpha f = \frac{\partial}{\partial x} \frac{\partial^3}{\partial y^3} f = 12x \exp(x^2). \]
We define by $C^k(\Omega)$ as the set of functions with continuous derivatives of order less than or equal to $k$ in the open set $\Omega$.

We define the support of a function as the closure

$$\text{supp}[f] := \{ x \mid f(x) \neq 0 \}.$$

Then, we define the set of compactly supported functions with derivatives up to order $k$ over the set $\Omega$ as

$$C^k_c(\Omega) := \{ f \in C^k(\Omega) \mid \text{supp}[f] \text{ is a compact set} \}.$$

**Test functions.** Any element in $C^\infty_c(\Omega)$ is called a test function. For example, the mollifier

$$\varphi(t) = \begin{cases} \exp(1/t) & t < 0 \\ 0 & t \geq 0 \end{cases}.$$

A linear form $u$ on $C^\infty_c(\Omega)$ satisfies

$$\langle u, a_1 \varphi_1 + a_2 \varphi_2 \rangle = a_1 \langle u, \varphi_1 \rangle + a_2 \langle u, \varphi_2 \rangle$$

for scalars $a_1, a_2$ and functions $\varphi_1, \varphi_2 \in C^\infty_c(\Omega)$.

**Distribution.** A linear form $u$ on $C^\infty_c(\Omega)$ is called a distribution if, for any compact set $K \subset \Omega$, there exist a constants $C \geq 0$ and $N \in \mathbb{N}$ such that

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \varphi|$$

for all $\varphi \in C^\infty_c(\Omega)$ with support in $K$.

### 2.2 Continuity of distributions

**Theorem 2.1.** Let $f$ be continuous on $\Omega$, and let

$$\langle f, \varphi \rangle = \int_\Omega f \varphi \, dx$$

for all $\varphi \in C^\infty_c(\Omega)$. Then, $\langle f, \varphi \rangle$ is a distribution.

**Proof.** To see that $\langle f, \varphi \rangle$ defines a distribution see that

$$\langle f, a_1 \varphi_1 + a_2 \varphi_2 \rangle = a_1 \langle f, \varphi_1 \rangle + a_2 \langle f, \varphi_2 \rangle.$$
Moreover, for a compact set $K \subset \Omega$, and $\varphi \in C_c^\infty$ with $\text{supp}(\varphi) \subset K$ we have
\[
|\langle f, \varphi \rangle| = \left| \int_K f \varphi \, dx \right| \leq \sup_K |\varphi| \int_K |f| \, dx.
\]
By definition, it follows that $\langle f, \varphi \rangle$ is a distribution. \hfill \square

**Theorem 2.2.** Let $f$ be continuous on $\Omega$ then, if
\[
\langle f, \varphi \rangle = 0
\]
for all $\varphi \in C_c^\infty$, it follows that
\[
f \equiv 0
\]
in $\Omega$.

**Proof.** Assume by contradiction that there exists $y \in \Omega$ such that $f(y) \neq 0$. Then, by continuity of $f$ there exists $\delta$ such that $f(x) \geq 0$ whenever $|x - y| < \delta$ (without loss of generality we assumed $f(y) > 0$). Without giving the explicit form, there exist $\varphi \geq 0 \in C_c^\infty$ with $\text{supp}(\varphi) \subset \{ x \ | \ |x - y| < \delta \}$. Then,
\[
\int_\Omega f \varphi \, dx > 0,
\]
a contradiction to our hypothesis. \hfill \square

**Note.** For the remainder of the manuscript we simple use the notation $f$ to imply a distribution of the form $\langle f, \varphi \rangle$.

**Remark.** Equivalence of distributions is defined as
\[
f = g
\]
if and only if
\[
\langle f, \varphi \rangle = \langle g, \varphi \rangle
\]
for all $\varphi \in C_c^\infty(\Omega)$.

**Lemma 2.1.** The map $u : C(\Omega) \to \mathcal{D}'(\Omega)$ given by $f \mapsto \langle f, \varphi \rangle$ is a bijection.

**Lemma 2.2.** Let $f$ be a locally integrable function; that is, $f \in L^1_{\text{loc}}(\Omega)$. Then, $\langle f, \varphi \rangle$ is a distribution.

**Proof.** Linearity is inherited from the definition of integrals. Then, for a compact set $K$
\[
|\langle f, \varphi \rangle| \leq \sup_K |\varphi| \int_K |f| \, dx < \infty
\]
\hfill \square

**Example 2.2.1.** (Dirac-delta distribution) Let $\delta$ be the Dirac-delta function. Then,
\[
\langle \delta, \varphi \rangle = \varphi(0)
\]
Remark. We say that the sequence $\varphi_j \in C^\infty_c(\Omega)$ converges to 0 if $\text{supp}(\varphi) \subset \Omega$, and for all $\alpha$, $\partial^\alpha \varphi_j \to 0$ uniformly as $j \to \infty$.

2.3 Convergence of distributions

Let $u_n$ be a sequence in $\mathcal{D}'(\Omega)$. We say that $u_n$ converges to $u \in \mathcal{D}'(\Omega)$ if

$$\langle u_n, \varphi \rangle \to \langle u, \varphi \rangle$$

as $n \to \infty$, for all $\varphi \in C^\infty_c(\Omega)$.

Example. Let $f_n = e^{inx}$. Note that $f_n$ does not converge in the classical sense as $n \to \infty$. However, as a distribution we have (by integration by parts)

$$\langle f_n, \varphi \rangle = \int_{\Omega} e^{inx} \varphi(x) \, dx = -\frac{1}{in} \int_{\Omega} e^{inx} \varphi'(x) \, dx.$$  

Note that the latter converges to 0. Thus, as a distribution, $f_n \to 0$.

Example. Consider a continuous function $f$ so that $\text{supp}(f) \subset [0, 1]$ and $\int f = 1$. Then, let $f_k(x) = kf(kx)$ with $k \in \mathbb{Z}^+$. These are refer to as “tent functions”. In the distributional sense

$$\langle f_k, \varphi \rangle = \int f_k(x) \varphi(x) \, dx = \int kf(kx) \varphi(x) \, dx = \int f(y) \varphi(y/k) \, dy.$$  

Then,

$$|\langle f_k, \varphi \rangle - \varphi(0)| \leq \int |f(y)| |\varphi(y/k) - \varphi(0)| \, dy \leq \sup_{|z| \leq 1/k} |\varphi(z) - \varphi(0)| \int |f| \, dy.$$  

Note that the latter implies that $\langle f, \varphi \rangle \to \varphi(0)$ as $k \to \infty$. Thus, as a distribution

$$f \to \delta,$$

where $\delta$ is the Dirac-delta distribution.

We are interested in answering the question. When does the sequence $f_n$ converges to $f$ both in the classical, and in the distribution sense? To answer this question, we introduce the following theorem

Theorem 2.3 (Lebesgue Dominated Convergence). Let the sequence $f_n$ satisfy the following:

i. For all $n$, $f_n$ is an integrable function.

ii. The sequence $f_n$ converges to $f$ almost everywhere.
iii. There exist an integrable function $g$ such that, for all $n$, $f_n \leq g$. Then,

$$\int f_n \to \int f$$

as $n \to \infty$.

**Theorem 2.4.** The sequence $f_n$ converges to $f$ pointwise, and in the distribution sense if

i. The sequence $f_n$ is locally integrable,

ii. $f_n$ converges to $f$ almost everywhere, and

iii. there exist a locally integrable function $g$ such that, $|f_n| < g$, for all $n$.

**Proof.** Let $\varphi \in C^\infty_c(\Omega)$. Consider the sequence $\langle f_n, \varphi \rangle$. Note that, as a function, $f_n(x)\varphi(x)$ satisfies the criteria of theorem 2.3. Thus

$$\int f_n(x)\varphi(x) \, dx = \int f(x)\varphi(x) \, dx$$

as we wanted. \qed

### 2.4 Calculus of distributions.

Let $u \in C^1(\Omega)$ and let $\varphi \in C^\infty_c$ note that

$$\langle \partial_j u, \varphi \rangle = \int_\Omega (\partial_j u) \varphi \, dx = \left| u\varphi \right|_{\partial\Omega} - \int_\Omega u\partial_j \varphi \, dx. \quad (8)$$

Thus,

$$\langle \partial_j u, \varphi \rangle = -\langle u, \partial_j \varphi \rangle \quad (9)$$

**Definition 2.1** (Differentiation). Let $u \in D'(\Omega)$ then, the distribution $\partial_j u$ satisfies

$$\langle \partial_j u, \varphi \rangle = -\langle u, \partial_j \varphi \rangle$$

for all $\varphi \in C^\infty_c$.

**Theorem 2.5.** Let $u_n$ be a sequence of distributions such that $u_n \to u$ as $n \to \infty$, with $u \in D'(\Omega)$. Then,

$$\partial^\alpha u_n \to \partial^\alpha u$$

in $D'(\Omega)$. Note that is argument is not true in the classical sense.

**Proof.** Recall that there exist $C$ and $N$ such that

$$|\langle u, \varphi \rangle| \leq C \sum_{|\beta| \leq N} \sup_K |\partial^\alpha \varphi|.$$
Then,
\[ \langle \partial^\alpha u_n, \varphi \rangle = (-1)^\alpha \langle u_n, \partial^\alpha \varphi \rangle. \]
In the limit, we get
\[ (-1)^\alpha \langle u_n, \partial^\alpha \varphi \rangle = (-1)^\alpha \langle u, \partial^\alpha \varphi \rangle = \langle \partial^\alpha u_n, \varphi \rangle \]
as we wanted.

We are interested in functions whose distributional derivative is equal to its classical derivative.

**Theorem 2.6.** If \( \partial_j u \) exists and is continuous, then its classical derivative; namely
\[ u'(x) = \lim_{h \to 0} \frac{u(x + he_j) - u(x)}{h} \]
coincides with the distributional derivative of \( u \).

**Proof.** Let \( g(x) \) be the classical derivative of \( u(x) \). Note that, on the other hand, the distributional derivative of \( u \) is of the form
\[ -\int u(x) \varphi'(x) dx. \]
Now, since \( \partial_j u \) exists and is continuous, we can apply integration by parts in the expression above to get
\[ -\int w \varphi' = \int g \varphi = \langle g, \varphi \rangle \]
as we wanted to show. \( \square \)

### 2.5 Products on distributions

Note that, in the usual sense, the point-wise product of smooth functions is also a smooth function. However, the product of two distributions is not well defined. Thus, we define

**Definition 2.2** (The product of a distribution with a smooth function). Let \( u \in \mathcal{D}'(\Omega) \) and \( m \in C^\infty(\Omega) \). The product \( mu \) is defined as
\[ \langle mu, \varphi \rangle = \langle u, m\varphi \rangle. \]
Where \( \varphi \in C_c^\infty(\Omega) \)

Now, for a given distribution \( v \in \mathcal{D}'(\Omega) \) we wish to find \( u \) such that
\[ u' = v. \]
Note that  
\[ \langle u', \varphi \rangle = -\langle u, \varphi' \rangle = \langle v, \varphi \rangle \]
To define \( \varphi \) as \( \int \psi \) is not correct as integration does not preserve \( C^\infty_c \) without some restrictions to \( \psi \).

**Lemma 2.3.** Let \( g \) and \( G \) be in \( C^\infty_c(\mathbb{R}) \)

1. If \( G' = g \) then, \( \langle 1, g \rangle = 0 \).

2. If \( \langle 1, g \rangle = 0 \) then, there exist \( G \in C^\infty_c \) such that \( G = \int g \).

Let \( \varphi_0 \in C^\infty_c(\mathbb{R}) \) such that \( \int \varphi_0 = 1 \). Then, for all \( \varphi \in C^\infty_c(\mathbb{R}) \)

\[ \varphi = (\varphi - \langle 1, \varphi \rangle \varphi_0) + \langle 1, \varphi \rangle \varphi_0. \]

Now see that

\[ \langle 1, \varphi - \langle 1, \varphi \rangle \varphi_0 \rangle = \langle 1, \varphi \rangle - \langle 1, \varphi \rangle = 0. \]

Therefore, by lemma 2.3, there exist \( \psi(x) \), such that

\[ \psi(x) = \int_{-\infty}^{x} \varphi(t) - \langle 1, \varphi \rangle \varphi_0(t) \, dt \in C^\infty_c(\mathbb{R}). \]  

**Theorem 2.7.** Let \( u \in D'(\Omega) \) with \( u' = 0 \). Then, \( u \) is a constant distribution.

**Proof.** For any \( \varphi \in C^\infty_c \), let \( \psi \) be as in (10), then

\[ \langle u', \psi \rangle = \langle 0, \psi \rangle. \]

On the other hand

\[ \langle u', \psi \rangle = -\langle u, \psi' \rangle = -\langle u, \varphi \rangle + \langle \langle u, \varphi_0 \rangle, \varphi \rangle. \]

It follows that

\[ 0 = -\langle u, \varphi \rangle + \langle \langle u, \varphi_0 \rangle, \varphi \rangle, \]

then,

\[ \langle u, \varphi \rangle = \langle \langle u, \varphi_0 \rangle, \varphi \rangle. \]

Note that \( \varphi_0 \) was arbitrarily chosen, thus \( u \) is constant for all \( \varphi \in C^\infty_c(\Omega) \).

\[ \square \]