Abstract

What follows is a potpurri of notes and problems as discussed during the Fall semester at the University of Minnesota in the course M8402 Modeling and Applied Mathematics. The author is responsible for the solutions presented.

E1-1. Consider the reaction diffusion equation satisfied by the chemical concentration $c(x, t)$ inside a three-dimensional ball of radius $R$:

$$\frac{\partial c}{\partial t} = D \Delta c - kc, \text{ in } |x| < R,$$
$$c = c_*(2 + \cos(\omega t)), \text{ on } |x| = R,$$

where $D, k, R, c_*$ and $\omega$ are positive constants. Make the above system dimensionless, and identify the dimensionless constant(s).

Solution. Let the change of variables be:

$$c = c_* \hat{c}, \quad x = R \hat{x}, \quad t = \frac{1}{\hat{k}} \hat{t}.$$

Then, our system becomes:

$$\begin{cases} \frac{k c_*}{\frac{\partial}{\partial t}} \hat{c} = \frac{D}{R^2} c_* \Delta \hat{c} - k c_* \hat{c} \\ c_* \hat{c} = c_* \left(2 + \cos \left(\frac{\omega}{\hat{k}} \hat{t}\right)\right) \end{cases}$$

Upon simplification, the above is equivalent to

$$\begin{cases} \frac{\partial}{\partial t} \hat{c} = \alpha \Delta \hat{c} - \hat{c} \\ \hat{c} = 2 + \cos \beta \hat{t} \end{cases}$$

where $\alpha = DR^{-2}k^{-1}$, and $\beta = \omega k^{-1}$. Note that the system is now dimensionless. \hfill \Box

E1-2. Find the leading order solution to the following equation:

$$\varepsilon y'' - y' - 2xy^2 = 0, \quad y(0) = 1, y'(1) = 0,$$

where $\varepsilon$ is small and positive (No need to find composite expansion, just the leading order solution is fine)
**Solution.** Assume \( y \) has the form:

\[
y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots.
\]

Then, by substitution, we have that:

\[
\varepsilon \left( y_0'' + \varepsilon y_1'' + \cdots \right) - \left( y_0' + \varepsilon y_1' + \cdots \right) - 2x \left( y_0 + \varepsilon y_1 + \cdots \right)^2 = 0.
\]

Then, the leading order term \( O(1) \) satisfies:

\[
-y_0' - 2x y_0^2 = 0.
\]

Note that the equation is separable, with solution

\[
y_0 = \frac{1}{x^2 + C}.
\]

The above cannot satisfy the boundary condition \( y(1) = 0 \). Thus, we presume the existence of a boundary layer at \( x = 1 \). That aside, we compute the value of \( C \) with the constraint \( y(0) = 1 \), which leads to \( C = 1 \). We proceed to inspect the boundary layer, let \( \tilde{x} = (x - 1)/\varepsilon \), and \( y(x) = \tilde{y}(\tilde{x}) \), substitution into the initial problem yields:

\[
\varepsilon^{1-2\alpha} \tilde{y}'' - \varepsilon^{-\alpha} \tilde{y}' - 2(\varepsilon^\alpha \tilde{x} + 1) \tilde{y} = 0.
\]

In order to preserve a second order ODE we balance the terms with \( \alpha = 1 \), so that:

\[
\varepsilon^{-1} \tilde{y}'' - \varepsilon^{-1} \tilde{y}' - 2(\varepsilon \tilde{x} + 1) \tilde{y} = 0.
\]

The leading order term \( O(\varepsilon^{-1}) \) satisfies:

\[
\tilde{y}'' - \tilde{y}' = 0,
\]

and therefore,

\[
\tilde{y} = C_1 + C_2 \varepsilon \tilde{x}.
\]

Imposing the boundary layer condition \( y(1) = 0 \), or equivalently, \( \tilde{y}(0) = 0 \) we have that \( C_2 = -C_1 \). Thus, \( y(\tilde{x}) = C[1 - \exp(\tilde{x})] \), and moreover, matching the boundary layers:

\[
\lim_{\tilde{x} \to -\infty} C(1 - e^{\tilde{x}}) = \lim_{x \to 1} \frac{1}{x^2 + 1},
\]

we get \( C = 1/2 \). We conclude that the leading order expansion is of the form:

\[
y_0 = \frac{1}{x^2 + 1} + \frac{1}{2} \left( 1 - e^{(x-1)/\varepsilon} \right) - \frac{1}{2}.
\]
E1-3. Consider the following problem for \( u(x, t) \):

\[
\frac{\partial u}{\partial t} + (1 - 2u) \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \varepsilon > 0, x \in \mathbb{R}, t > 0,
\]

\[u(x, 0) = \phi(x).\]

In the above, \( \varepsilon > 0 \) is assumed to be small.

(a) Let \( \phi(x) \) be a continuous function. For certain kinds of initial data, we expect the formation of shocks in finite time for small \( \varepsilon \). Explain what kind of initial data, and explain your reasoning.

**Solution.** Let \( v = 1 - 2u \), and \( \varphi(x) = (1 - \nu(x, 0))/2 = u(x, 0) \). Note that \( \nu_t = -2u_t \), \( \nu_x = -2u_x \), and \( \nu_{xx} = -2u_{xx} \). Thus, our problem is equivalent to the classical Burger’s equation:

\[
\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \varepsilon > 0, x \in \mathbb{R}, t > 0,
\]

\[v(x, 0) = \psi(x).\]

We let \( v \) have asymptotic expansion:

\[v(x, t) = v_0(x, t) + \varepsilon v_1(x, t) + \varepsilon^2 v_2(x, t) + \cdots\]

The first order term \( O(1) \) satisfies:

\[\frac{\partial v_0}{\partial t} + v_0 \frac{\partial v_0}{\partial x} = 0.\]

Using the method of characteristic lines, we find that the solution is constant along the curves:

\[x = x_0 + \psi(x_0) t.\]

Moreover, the solution of \( v \) from its characteristics is well defined only if the curves do not intersect and the cover the upper half plane. Thus, shock waves occur if \( \psi'(x_0) < 0 \). Since \( \psi'(x_0) = v'(x_0, 0) = -2u'(x_0, 0) = 1/2 \phi(x_0) \). Our initial problem has shock waves whenever

\[\phi'(x_0) > 0.\]

\( \square \)

(b) Let \( x = s(t) \) be a shock location at time \( t \). Use matched asymptotic to find the velocity of the shock in terms of the value of \( u \) just to the left and right of the shock location.

**Solution.** Let \( \xi = (x - s(t))/\varepsilon^a \), and let \( u \) be a function of \( \xi \). Then,

\[
\frac{\partial u}{\partial t} = -\frac{1}{\varepsilon^a} s'(t) \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial t}.
\]
and our equation has the form:

\[-\varepsilon^{-\alpha} s'(t) \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial t} + \varepsilon^{-\alpha} (1 - 2u) \frac{\partial u}{\partial \xi} = \varepsilon^{1-2\alpha} \frac{\partial^2 u}{\partial \xi^2}.\]

Form here we set \(\alpha = 1\) as to balance the terms \(u_{\xi}\) and \(u_{\xi \xi}\). Our leading order term \(O(\varepsilon^{-1})\) satisfies:

\[
(s'(t) - 1)u_{\xi} + 2uu_{\xi} = u_{\xi \xi}.
\]

Note that the above is equivalent to:

\[
\frac{\partial}{\partial \xi} (s'(t) - 1)u + u^2 = \frac{\partial}{\partial \xi} u_{\xi},
\]

thus, by integrating against \(\xi\) our equation satisfies:

\[
(s'(t) - 1)u + u^2 = u_{\xi}.
\]

Recall that our problem has to satisfy the following boundary layer conditions

\[
\lim_{\xi \to \infty} u = u^+, \quad \lim_{\xi \to -\infty} u = u^-, \quad \lim_{\xi \to |\infty|} u_{\xi} = 0.
\]

It follows that:

\[
\lim_{\xi \to \infty} (s'(t) - 1)u + u^2 = (s'(t) - 1)u^+ + (u^+)^2 = 0
\]

and,

\[
\lim_{\xi \to -\infty} (s'(t) - 1)u + u^2 = (s'(t) - 1)u^- + (u^-)^2 = 0.
\]

Setting the above equal to each other, and after simplification, we get:

\[
\begin{align*}
    s'(t) &= -\frac{(u^+)^2 - (u^-)^2}{u^+ - u^-} + 1 \\
    &= -(u^+ + u^-) + 1.
\end{align*}
\]

\(\Box\)

**E1-4.** Let \(A\) be a \(n \times n\) matrix where \(n \geq 2\). Suppose \(A\) can be diagonalized, and \(\lambda_0 \in \mathbb{C}\) is an eigenvalue of \(A\) with a one-dimensional eigenspace. Consider the eigenvalue perturbation problem:

\[
(A + \varepsilon B)u = \lambda u,
\]

where \(B\) is some \(n \times n\) matrix and \(\varepsilon\) is small. We are interested in how the eigenvalue \(\lambda\) deviates from \(\lambda_0\) for small \(\varepsilon\).

(a) Let \(u_0\) be an eigenvector of \(A\) at \(\lambda_0\). Using the fact that \(A\) is diagonalizable (that is to say, \(A\) has a linearly independent set of eigenvectors), show that there is no vector \(v\) that satisfies \((A - \lambda_0 I)v = u_0\).

**Proof.** We proceed by contradiction. Assume that such vector \(v\) exists, and that \(A\) has a
set of linearly independent vectors $u_i$ with corresponding eigenvalues $\lambda_i$ that span $\mathbb{R}^n$. Thus, $v$ may be written as:

$$v = c_0 u_0 + c_1 u_1 + \cdots.$$ 

Without loss of generality, assume that $\lambda_0$ is the eigenvalue of $u_0$. Then, the vector $v$ satisfies

$$A(c_0 u_0 + c_1 u_1 + \cdots) - \lambda_0(c_0 u_0 + c_1 u_1 + \cdots) = u_0,$$

-equivalently,

$$c_0(\lambda_0 - \lambda_0)u_0 + c_1(\lambda_1 - \lambda_0)u_2 + c_3(\lambda_3 - \lambda_0)u_3 + \cdots = u_0.$$ 

Note that the first term in the left hand side is zero and therefore:

$$c_1(\lambda_1 - \lambda_0)u_2 + c_3(\lambda_3 - \lambda_0)u_3 + \cdots = u_0.$$ 

The above is a contradiction to the linear independence of the set $u_i$. It follows that $(A - \lambda_0 I)v = u_0$ has no solution, as desired.

\(\square\)

(b) Let $A^*$ be the adjoint of $A$ (complex conjugate transpose). Then, $\lambda_0$ is an eigenvalue of $A^*$. Using the result of the above problem, show that $\lambda_0$ eigenspace of $A$ and the $\lambda_0$ eigenspace of $A^*$ are not orthogonal to each other.

Proof. Assume that $u$ and $v$ are orthogonal. Then, $u$ is an element of $[\ker(A^* - \lambda_0 I)]^\perp$. By the Fredholm alternative the statement is equivalent to $u \in \text{Im}(A - \lambda_0 I)$. Thus, there exist $w$ in the space that satisfies:

$$(A - \lambda I)w = u,$$

A contradiction to part (a) in this problem.

\(\square\)

(c) Find the leading order correction to the eigenvalue $\lambda_0$ when $\epsilon \neq 0$. State clearly where you used the Fredholm alternative, and discuss how to find the leading order correction to the eigenvector.

Proof. Using the standard asymptotic expansions:

$$u = u_0 + \epsilon u_1 + \cdots,$$

and

$$\lambda = \lambda_0 + \epsilon \lambda_1 + \cdots$$

we arrive at the following $\epsilon$-order expressions:

$$O(1): \quad (A - \lambda_0 I)u_0 = 0$$

$$O(\epsilon): \quad (A - \lambda_1 I)u_1 = (\lambda_1 I - B)u_0.$$ 

We know that the $O(1)$ term has a solution by construction. Then, for the $O(\epsilon)$ term we want to find $w = (\lambda_1 I - B)u_0$. To that end, note that $w$ is an element of the image of $(A - \lambda_1 I)$.
$\lambda_0 I)$. Then, invoking the \textbf{Fredholm alternative}, $w$ is an element of the space $[\ker(A^* - \lambda_0 I)]^\perp$. Let $w'$ be an element of $\ker(A^* - \tilde{\lambda} I)$, we proceed to take the dot product of the $O(\varepsilon)$ term with respect to $w'$. We get, upon simple algebraic manipulations:

$$\lambda_1 = \frac{\langle Bu_0, w' \rangle}{\langle u_0, w' \rangle}.$$ 

Note that part (b) of the problem guarantees that $\langle u_0, w' \rangle \neq 0$. \hfill $\square$

\textbf{H1-1.} Consider the mass spring problem:

$$m\frac{d^2 y}{dt^2} + \gamma \frac{dy}{dt} + ky = 0, \quad y(0) = y_*, \quad \frac{dy}{dt}(0) = v_.*$$

a) Make the equation dimensionless.

\textit{Solution.} Our non-dimensionalization will depend on the parameter we expect to neglect. We begin by scaling the initial position and initial velocity with the maps,

$$y \mapsto \frac{y}{y_*}, \quad y' \mapsto \frac{y'}{v_*}.$$ 

Then, after simplification, the equation becomes:

$$m\frac{d^2 y}{dt^2} + \gamma \frac{dy}{dt} + ky = 0, \quad y(0) = 1, \quad \frac{dy}{dt}(0) = 1.$$ 

Next, we scale the time variable. Let $t \mapsto \sqrt{\frac{k}{m}} t$ and observe that the substitution induces the following maps:

$$y' \mapsto \sqrt{\frac{k}{m}} y', \quad y'' \mapsto \frac{k}{m} y''.$$ 

After substitution and simplification we get:

$$\frac{d^2 y}{dt^2} + \frac{x}{\sqrt{mk}} \frac{dy}{dt} + y = 0, \quad y(0) = 1, \quad \frac{dy}{dt}(0) = 1.$$ 

Note that the system is now dimensionless.
Alternatively, we may chose to scale the time variable by $t \rightarrow \frac{k}{t}$. Inducing the maps:

\[ y' \rightarrow \frac{k}{\gamma} y' \]
\[ y'' \rightarrow \left(\frac{k}{\gamma}\right)^2 y''. \]

After substitution and simplification, we arrive at the dimensionless system:

\[ \frac{mk}{\gamma^2} \frac{d^2 y}{dt^2} + \frac{dy}{dt} + y = 0, \quad y(0) = 1, \quad \frac{dy}{dt}(0) = 1. \]

b) Identify the small parameter that allows one to neglect the friction term $\gamma \frac{dy}{dt}$.

\textit{Answer.} Looking at the first non-dimensionalization of the system above, the parameter is

\[ \frac{\gamma}{\sqrt{mk}} \]

c) Identify the small parameter that allows one to neglect the inertial term $md^2 \frac{y}{dt^2}$.

\textit{Answer.} Again, looking at part a) the parameter is

\[ \frac{mk}{\gamma^2} \]

d) When the inertial term is neglected, the equation is a first order ODE and the initial conditions cannot be satisfied. Explain what is happening.

\textit{Explanation.} Indeed, if we ignore the inertial term, we are left (in the nondimensional case) with the ODE:

\[ \frac{dy}{dt} + y = 0, \quad y(0) = 1, \quad \frac{dy}{dt}(0) = 1. \]

However, the solution $y = C\exp(-x)$ can only satisfy one of the initial conditions.

\textbf{H1-3.} In the enzyme kinetics problem, suppose the chemical reaction to be considered is:

\[ 2S + E \leftrightarrow C \rightarrow E + 2P \]

Write down a suitable set of equations for each one and perform a similar quasi-steady state analysis in each of these cases.
Solution. Using the law of mass action kinetics we describe the system as follows:

\[
\begin{align*}
\frac{dS}{dt} &= -k_1ES^2 + k_2C \\
\frac{dE}{dt} &= -k_1ES^2 + (k_3 + k_2)C \\
\frac{dC}{dt} &= k_1ES^2 - (k_2 + k_3)C \\
\frac{dP}{dt} &= k_3C
\end{align*}
\]

with initial conditions:

\[
S(0) = s_*, \quad E(0) = e_*, \quad C(0) = 0, \quad P(0) = 0.
\]

We begin by scaling the variables as follows,

\[
S = s_\ast \hat{S}, \quad E = e_\ast \hat{E}, \quad P = s_\ast \hat{P}, \quad t = t_\ast \hat{t}.
\]

Moreover, observe that the molecule \(E\) exists as a bounded or free molecule, thus, the quantity \(E + C\) must remain constant with the value dictated by the initial conditions; i.e.

\[
E + C = e_\ast.
\]

Making the appropriate substitutions into the system and dropping the “hat” notation to avoid clutter we have the following system

\[
\begin{align*}
\frac{dS}{dt} &= -ES^2 + \frac{k_2}{k_1s_\ast^2}(1 - E) \\
\frac{e_\ast}{s_\ast} \frac{dE}{dt} &= -ES^2 + \frac{k_2 + k_3}{k_1s_\ast^2}(1 - E) \\
\frac{dP}{dt} &= \frac{k_3}{k_1s_\ast^2}(1 - E) \\
C &= e_\ast(1 - E)
\end{align*}
\]

together with the initial conditions

\[
S(0) = 1, \quad E(0) = 1, \quad C(0) = 0, \quad P(0) = 0.
\]

By letting \(\varepsilon = e_\ast/s_\ast\), \(\lambda = k_2/k_1s_\ast^2\) and \(\kappa = (k_2 + k_3)/(k_1s_\ast^2)\), we arrive at the non-dimensional
system:
\[
\begin{align*}
\frac{dS}{dt} &= -ES^2 + \lambda (1 - E) \\
\frac{dE}{dt} &= -ES^2 + \kappa (1 - E) \\
\frac{dP}{dt} &= (\kappa - \lambda) (1 - E) \\
C &= e^* (1 - E)
\end{align*}
\]

with initial conditions,
\[
S(0) = 1, \quad E(0) = 1, \quad C(0) = 0, \quad P(0) = 0.
\]

Now we assume that the ratio between enzyme and substrate is small; i.e. \(\varepsilon \approx 0\), this leads to the simplification \(-ES^2 + \kappa (1 - E) = 0\) or, equivalently,
\[
E = \frac{\kappa}{S^2 + \kappa}.
\]

However, note that at this point we no longer satisfy the initial condition \(E(0) = 1\). Moreover, observe that substituting \(E\) back to the system we have
\[
\frac{dS}{dt} = \frac{dP}{dt} = \frac{(\lambda - \kappa)S^2}{S^2 + \kappa}.
\]

Thus, we have reduced the system to a single ODE. \(\square\)

**H1-5.** Energy conservation can be used to derive equations for heat conduction. Suppose we have a region \(\Omega\) and let \(e(x)\) be the energy density of the system.

a) Using the same idea as in mass conservation, derive the equation:
\[
\frac{\partial e}{\partial t} + \nabla \cdot q = h
\]

where \(q\) is the heat flux and \(h\) is the heat source.

**Proof.** Let \(\omega\) be a region inside \(\Omega\). Then, the change of energy inside the region \(\omega\) can be expressed as the amount of energy (or heat) coming into \(\omega\) from a source \((h)\), and leaving around the boundary \(\partial \omega\) at the rate of the flux. Thus, equation thus reads,
\[
\frac{d}{dt} \int_{\omega} e(x, t) \, dx = \int_{\omega} h(x, t) \, dx - \int_{\partial \omega} q(x, t) \cdot n \, ds.
\]

Assuming smoothness of the equations we may interchange the integral and derivative from the first term and use the divergence theorem on the line integral, so that we have
\[
\int_{\omega} \frac{\partial}{\partial t} e(x, t) \, dx = \int_{\omega} h(x, t) \, dx - \int_{\omega} \nabla \cdot q(x, t) \, dx.
\]
Equivalently,
\[ \int_\omega \frac{\partial}{\partial t} e(x, t) + \nabla \cdot q(x, t) \, dx = \int_\omega h(x, t) \, dx. \]

Since \( \omega \) was chosen arbitrarily inside \( \Omega \) we get the desired relation
\[ \frac{\partial}{\partial t} e(x, t) + \nabla \cdot q(x, t) = h(x, t). \]

b) Suppose the energy density \( e = C_V T \) and \( q = -k \nabla T \). \( C_V \) is called the heat capacity and \( k \) the heat conductivity. Check that the temperature satisfies the “diffusion” equation. This is the reason why the “diffusion equation” and the “heat equation” are often synonymous.

**Proof.** We plugin the given relations in the equation above.
\[ \frac{\partial}{\partial t} C_V T(x, t) + \nabla \cdot (-k \nabla T(x, t)) = h(x, t). \]

Upon simplification we obtain
\[ C_V \frac{\partial}{\partial t} T(x, t) - k \nabla \cdot (\nabla T(x, t)) = h(x, t), \]
equivalently,
\[ \frac{\partial}{\partial t} T = \frac{-k}{C_V} \Delta T + h \]
is the heat equation. \( \square \)

H2-3. Consider the following eigenvalue problem:
\[ -\frac{\partial^2 u}{\partial x^2} + \varepsilon f(x) u = \lambda u, \quad n \in \mathbb{R}/2\pi \mathbb{Z}, \]
where \( f \) is a smooth periodic function. Here, \( \lambda \) is an eigenvalue and \( u \) is the eigenfunction.

(a) Find all eigenvalues and eigenfunctions of the above problem when \( \varepsilon = 0 \).

**Solution.** We make the assumption that \( u \) satisfies Dirichlet boundary conditions; i.e:
\[ u(0) = u(2\pi) = 0. \]

Then, for \( \varepsilon = 0 \) and \( u \in \mathbb{R}/2\pi \mathbb{Z} \) we have the eigenvalue problem:
\[ -\frac{d^2 u}{dx^2} = \lambda u, \quad u(0) = u(2\pi) = 0. \]
The above is a classic ODE problem with nontrivial solutions for \( \lambda > 0 \). Making the assumption that \( \lambda \) is positive we obtain the family of solutions:

\[
    u_n = \frac{1}{\sqrt{\pi}} \sin(\sqrt{\lambda_n}x)
\]

with \( \lambda_n = n^2 \). The normalization coefficient \( 1/\sqrt{\pi} \) will be useful when considering the perturbed problem.

(b) Discuss what happens with the eigenvalues and eigenfunctions when \( \varepsilon \neq 0 \) but small.

**Solution.** First, we prove the following property of the operator \( \frac{\partial^2}{\partial x^2} \).

**Useful Property.** For \( u \) and \( v \) satisfying zero boundary conditions on the interval \([0,2\pi]\) we have

\[
    \int_0^{2\pi} \frac{\partial^2 u}{\partial x^2} v \, dx = \int_0^{2\pi} u \frac{\partial^2 v}{\partial x^2} \, dx.
\]

In other words the operator \( \partial_{xx} \) is self-adjoint.

**Proof.** If we take integration by parts twice we get:

\[
    \int_0^{2\pi} \frac{\partial^2 u}{\partial x^2} v \, dx = \bigg| \frac{\partial}{\partial x} u \bigg|_0^{2\pi} v - \bigg| \frac{\partial}{\partial x} v \bigg|_0^{2\pi} u + \int_0^{2\pi} \frac{\partial^2 v}{\partial x^2} \, dx,
\]

Since both \( u \) and \( v \) are zero at the boundary, the intermediate values vanish and we are left with the desired result.

We presume that the eigenvalues and eigenfunctions of the problem have a regular expansion, namely, we let:

\[
    u_n = u_n^0 + \varepsilon u_n^1 + \mathcal{O}(\varepsilon^2)
\]

\[
    \lambda_n = \lambda_n^0 + \varepsilon \lambda_n^1 + \mathcal{O}(\varepsilon^2).
\]

Substitution into the original problem leads to

\[
    -\frac{\partial^2 u_n^0}{\partial x^2} + \varepsilon \frac{\partial^2 u_n^1}{\partial x^2} + \varepsilon f \cdot (u_n^0 + \varepsilon u_n^1) = (\lambda_n^0 + \varepsilon \lambda_n^1)(u_n^0 + \varepsilon u_n^1).
\]

Collection of coefficients yields the following problems:

- Of order \( \mathcal{O}(1) \)

\[
    -\frac{\partial^2 u_n^0}{\partial x^2} = \lambda_n^0 u_n^0
\]

whose solution was settled in part (a).
• Of order $O(\varepsilon)$

\[-\frac{\partial^2 u^1_n}{\partial x^2} + f(x)u^0_n = \lambda^0_n u^1_n + \lambda^1_n u^0_n\]

In this case we multiply the equation by $u^0_n$ and integrate from 0 to $2\pi$, note that this is the inner product in $L^2(0, 2\pi)$. Using standard notation and by making use of the useful property together with the orthonormal property of the eigenfunctions $u^0_n$ (from (a)) we get the following:

\[
\langle -\llangle u^1_n, u^0_n \rrangle + \langle f u^0_n, u^0_n \rangle = \lambda^0_n \langle u^1_n, u^0_n \rangle + \lambda^1_n \|(u^0_n)\|
\]

\[
\Rightarrow \langle u^1_n, u^0_n \rangle + \langle f u^0_n, u^0_n \rangle = \lambda^0_n \langle u^1_n, u^0_n \rangle + \lambda^1_n \|(u^0_n)\|
\]

\[
\Rightarrow \lambda^0_n \langle u^1_n, u^0_n \rangle + \langle f u^0_n, u^0_n \rangle = \lambda^0_n \langle u^1_n, u^0_n \rangle + \lambda^1_n \|
\]

\[
\Rightarrow \langle f u^0_n, u^0_n \rangle = \lambda^1_n
\]

equivalently,

\[
\lambda^1_n = \int_0^{2\pi} f(x)(u^0_n)^2 dx = \frac{1}{\pi} \int_0^{2\pi} f(x)\sin^2(nx) dx.
\]

Note that $f(x)$ is smooth function thus, the integral is well defined. Now we are left to find $u^1_n$. By the orthogonality of our eigenfunction we may assume that both $u^1_n$ and $f(x)$ have an expansion of the form

\[
u^1_n = \sum_{j=1}^{\infty} a_n ju^0_j
\]

\[
f(x) = \sum_{j=1}^{\infty} g_n ju^0_j.
\]

Substitution into the original systems leads to:

\[a_{nk} = \frac{gnk}{\lambda^0_n - \lambda^0_k}\]

for $k \neq n$.

\[
H3-1. \text{ (From Holmes 2.1) } \text{The Friedrich model problem for a boundary layer in a viscous fluid is (Friedrichs, 1941):}
\]

\[\varepsilon y'' = a - y',\]

for $0 < x < 1$, where $y(0) = 0$, $y(1) = 1$, and $a$ is a given positive constant with $a \neq 1$.

(a) After finding the first term of the inner and outer expansions, derive a composite expansion of the solution of this problem.
(b) Derive a two-term composite expansion of the solution of this problem.

Solution. Let $y$ have an expansion of the form:

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

Substitution into the differential equation yields:

$$\varepsilon (y''_0 + \varepsilon y''_1 + \varepsilon^2 y''_2 + \cdots) = a - (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots).$$

Then, at leading order we have: $O(1)$:

$$a - y'_0 = 0,$$

with general solution $y + 0 = ax + A$.

Now, we assume that a boundary layer exists around $x = 0$, and we let $\hat{x} = x/\varepsilon^\alpha$. Let the solution near the boundary be

$$\hat{y}(\hat{x}) = \hat{y}_0 + \varepsilon \hat{y}_1 + \varepsilon^2 \hat{y}_2 + \cdots.$$

Substitution into the differential equation yields:

$$\varepsilon^{1-2\alpha} (\hat{y}''_0 + \varepsilon \hat{y}''_1 + \varepsilon^2 \hat{y}''_2 + \cdots) = a - \varepsilon^{-\alpha} (\hat{y}_0 + \varepsilon \hat{y}_1 + \varepsilon^2 \hat{y}_2 + \cdots).$$

The analysis of the term-wise order comparison leads to $\alpha = 1$ with leading order terms $O(\varepsilon^{-1})$:

$$\hat{y}''_0 = -\hat{y}'_0.$$

The solution of this differential equation is $\hat{y} = B + Ce^{-\hat{x}}$. Then, to fix the constants $A, B$ and $C$ we use the properties of the layer location, namely:

$$\left\{ \begin{array}{l}
\lim_{\hat{x}\to\infty} \hat{y}_0 = \lim_{x\to 0} y_0 \\
y_0(1) = 1 \\
\hat{y}_0(0) = 0
\end{array} \right.$$

These conditions lead to $A = C = -B = 1 - a$. Then we have:

$$y_0 = a(x - 1) + 1$$

$$\hat{y}_0 = (1 - a)(1 - e^{-x/\varepsilon}).$$

Written as a composite solution (and after simplification) we get

$$y(x) = ax + (1 - a)(1 - e^{-x/\varepsilon}).$$
H3-2 (From Holmes 2.2a) Find a composite expansion of the solution of:

\[ \varepsilon y'' + 2y' + y^3 = 0, \]

for \(0 < x < 1\), where \(y(0) = 0\) and \(y(1) = 1/2\).

Solution. As usual, assume that \(y(x)\) has expansion equal to:

\[ y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots. \]

Substitution into the differential equation yields:

\[ \varepsilon (e^{-\varepsilon x} y''_0 + \varepsilon y''_1 + \cdots) + 2(e^{-\varepsilon x} y'_0 + \varepsilon y'_1 + \cdots) + (y_0 + \varepsilon y_1 + \cdots)^3 = 0. \]

Trial and error teaches us that the boundary layer occurs near \(x = 0\). Thus, up to order \(O(1)\), the system satisfies:

\[ 2y'_0 + y_0^3 = 0, \]

with \(y_0(1) = 1/2\). The solution of this system yields:

\[ y_0(x) = \frac{1}{\sqrt{3} + x}. \]

Now, to solve the boundary layer problem we set \(\hat{x} = x/e^{\alpha}\), and we let \(\hat{y}(\hat{x})\) have expansion:

\[ \hat{y}(\hat{x}) = \hat{y}_0(\hat{x}) + \varepsilon \hat{y}_1(\hat{x}) + \varepsilon^2 \hat{y}_2(\hat{x}) + \cdots. \]

Substitution into the differential equation yields:

\[ \varepsilon^{1-2\alpha} (\hat{y}'_0'' + \varepsilon \hat{y}'_1'' + \cdots) + 2\varepsilon^{-\alpha} (\hat{y}'_0' + \varepsilon \hat{y}'_1' + \cdots) + (\hat{y}_0 + \varepsilon \hat{y}_1 + \cdots)^3 = 0. \]

Analysis of \(\varepsilon\)-powers yields \(\alpha = 1\). Then, our leading term, is of order \(O(e^{-1})\) is:

\[ \hat{y}'_0' + 2\hat{y}'_0 = 0, \]

with \(\hat{y}_0(0) = 0\). The solution is:

\[ \hat{y}(\hat{x}) = A(1 - e^{-2\hat{x}}). \]

Now we match our solutions, that is:

\[ \lim_{\hat{x} \to \infty} A(1 - e^{-2\hat{x}}) = \lim_{x \to 0} \frac{1}{\sqrt{3} + x}. \]

It is clear that \(A = 1/\sqrt{3}\). Thus, our composite solution is:

\[ y(x) = \frac{1}{\sqrt{3} + x} + \frac{1}{\sqrt{3}} \left(1 - e^{-2x/e}\right) - \frac{1}{\sqrt{3}}. \]

\(\square\)
H3-3. Find an approximation to leading order of the following problem in $\mathbb{R}^3$:

\begin{align*}
-\varepsilon \Delta u + u &= 1, \quad r < 1 \\
-\alpha \varepsilon \Delta v + v &= -1, \quad \alpha > 0, 1 < r < 2 \\
\frac{\partial u}{\partial r} &= \alpha \frac{\partial v}{\partial r}, \quad u - v = f, 2 < r < 1 \\
\frac{\partial v}{\partial r} &= 0, \quad r = 2
\end{align*}

where $r$ is the distance from the origin, $f < 2$ is a smooth function defined on the sphere $r = 1$, $\alpha > 0$ is a constant and $\varepsilon > 0$ is a small parameter.

Solution. Assume

\begin{align*}
u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \\
v &= v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots
\end{align*}

Substitution yields:

\begin{align*}
-\varepsilon \Delta (u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots) + (u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots) &= 1 \\
-\alpha \varepsilon \Delta (v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots) + (v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots) &= -1 \\
\frac{\partial}{\partial r} (u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots) &= \alpha \frac{\partial}{\partial r} (v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots) \\
\frac{\partial}{\partial r} (v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots) &= 0
\end{align*}

Thus, at leading order $O(1)$ we have:

\begin{align*}
u_0 &= 1, \quad r < 1 \\
v_0 &= -1, \quad 1 < r < 2.
\end{align*}

However, some of the boundary conditions are not satisfied. We proceed to inspect the boundary layer at $r = 1$ and $r = 2$. Starting with the case $r = 1$ we let:

$$\hat{r} = \frac{r - 1}{\varepsilon \alpha},$$

and

\begin{align*}
\hat{u}(\hat{r}) &= \hat{u}_0 + \varepsilon \hat{u}_1 + \varepsilon^2 \hat{u}_2 + \cdots \\
\hat{v}(\hat{r}) &= \hat{v}_0 + \varepsilon \hat{v}_1 + \varepsilon^2 \hat{v}_2 + \cdots
\end{align*}
where substitution (about $r = 1$) yields the system:

\[
\begin{align*}
-\varepsilon^{1-2\alpha} \Delta (\hat{u}_0 + \varepsilon \hat{u}_1 + \varepsilon^2 \hat{u}_2 + \cdots) + (\hat{u}_0 + \varepsilon \hat{u}_1 + \varepsilon^2 \hat{u}_2 + \cdots) &= 1 \\
-a\varepsilon^{1-2\alpha} \Delta (\hat{v}_0 + \varepsilon \hat{v}_1 + \varepsilon^2 \hat{v}_2 + \cdots) + (\hat{v}_0 + \varepsilon \hat{v}_1 + \varepsilon^2 \hat{v}_2 + \cdots) &= -1 \\
\varepsilon^{-\alpha} \frac{\partial}{\partial \hat{r}} (\hat{u}_0 + \varepsilon \hat{u}_1 + \varepsilon^2 \hat{u}_2 + \cdots) &= a\varepsilon^{-\alpha} \frac{\partial}{\partial \hat{r}} (\hat{v}_0 + \varepsilon \hat{v}_1 + \varepsilon^2 \hat{v}_2 + \cdots),
\end{align*}
\]

Life has taught us that, in this case, letting $\alpha = 1$ is more fruitful thus, at leading order $O(\varepsilon^{-1})$ we get:

\[
\begin{align*}
\Delta \hat{u}_0 &= 0 \\
-a\Delta \hat{v}_0 &= 0 \\
\frac{\partial}{\partial \hat{r}} \hat{u}_0 &= a\frac{\partial}{\partial \hat{r}} \hat{v}_0, \quad \hat{r} = 0.
\end{align*}
\]

If we assume the functions $u$ and $v$ are functions of $r$ only we may proceed to integrate, thus:

\[
\begin{align*}
\hat{u}_0 &= C_1 \hat{r} + C_2 \\
\hat{v}_0 &= C_3 \hat{r} + C_4 \\
\frac{\partial}{\partial \hat{r}} \hat{u}_0 &= a\frac{\partial}{\partial \hat{r}} \hat{v}_0, \quad \hat{r} = 0.
\end{align*}
\]

The latter condition fixes $C_1 = aC_3$, equivalently:

\[
\begin{align*}
\hat{u}_0 &= C_1 \hat{r} + C_2 \\
\hat{v}_0 &= \frac{C_1}{a} \hat{r} + C_3
\end{align*}
\]

We proceed to match the layers. Thus, we set:

\[
\begin{align*}
\lim_{\hat{r} \to 0} \hat{u} &= \lim_{r \to 1} u_0 \\
\lim_{\hat{r} \to 0} \hat{v} &= \lim_{r \to 1} v_0,
\end{align*}
\]

equivalently,

\[
\begin{align*}
C_2 &= 1 \\
C_3 &= -1
\end{align*}
\]

The last unknown constant is fixed by the condition $u - v = f < 2$. Lastly, we inspect the behavior around $r = 2$. A similar expansion, now with the change of coordinates:

\[
\hat{r} = \frac{r - 2}{\varepsilon^a},
\]

16
and keeping $\alpha = 1$, leads to:

\[-a\Delta \hat{v}_0 = 0\]
\[\frac{\partial}{\partial r} \hat{v}_0 = 0.\]

Just as before, we assume $\hat{v}_0$ is a function of $\hat{r}$ only and thus:

\[\hat{v}_0 = C_1 \hat{r} + C_2\]
\[C_1 = 0.\]

We conclude that around the boundary layer $r = 2$, our solution remains constant, trivially $\hat{v}_0 = -1$.

We conclude that, except for a boundary layer around $r = 1$, the solution behaves as a constant, whereas around $r = 1$ we have a layer that may be approximated by a linear function of slope $1/\epsilon$.

**H4-1.** The following problem is mostly a review of what we did in class. Consider

\[y'' + \epsilon f(y, y') + y = 0, \quad t > 0\]

where $\epsilon$ is small and $f$ is some smooth function.

(a) Use the multiple time scale expansion:

\[y = y_0(t_1, t_2) + \epsilon y_1(t_1, t_2) + \cdots, \quad t_1 = t, \quad t_2 = \epsilon t\]

and show that $y_0$ can be written as:

\[y_0 = A(t_2) \cos(\tau), \quad \tau = t_1 + \phi(t_2)\]

where $\phi(t_2)$ is some function of $t_2$.

**Solution.** We proceed with the suggested substitution. Note that

\[y'(u, v) = \sum_{k=0}^{\infty} \epsilon^k \left( \frac{\partial}{\partial u} y_k + \epsilon \frac{\partial}{\partial v} y_k \right)\]

and thus,

\[y''(u, v) = \sum_{k=0}^{\infty} \epsilon^k \left( \frac{\partial^2}{\partial u^2} y_k + 2\epsilon \frac{\partial^2}{\partial u \partial v} y_k + \epsilon^2 \frac{\partial^2}{\partial v^2} y_k \right)\]

Moreover, using the assumption that $f$ is a smooth function, we expand $f(y, y')$ about
\[ \varepsilon = 0. \] Hence,

\[
f(y, y') = f(y, y') \bigg|_{\varepsilon = 0} + \varepsilon \frac{d}{d\varepsilon} f(y, y') \bigg|_{\varepsilon = 0} + O(\varepsilon^2)
\]

\[
= f\left(y_0(u, v), \frac{\partial}{\partial u} y_0\right) + O(\varepsilon).
\]

With the above expansions, at leading order we get,

\[
\frac{\partial^2}{\partial u^2} y_0 + y_0 = 0.
\]

Assuming \( y_0 \) has the form \( U(u)V(v) \), for nonzero functions \( U(u) \) and \( V(v) \), we note that our equation satisfies:

\[
U'' V + UV = 0.
\]

Since we assume \( V \neq 0 \), \( U(u) \) has to satisfy the ODE \( U'' + U = 0 \). It should be clear that the solution is \( U(u) = a \sin(u) + b \cos(u) \). Thus, \( y_0 \) has the form:

\[
y_0 = A(v) \sin(u) + B(v) \cos(u),
\]

where \( A \) and \( B \) are functions of \( v \). Note that, equivalently, (using the phase-amplitude form of the solution) we get:

\[
y_0 = A(v) \cos(u + \varphi(v))
\]

as desired. \[ \square \]

(b) Show that the \( O(\varepsilon) \) equation is:

\[
\frac{\partial^2 y_1}{\partial t_1^2} + y_1 = 2\left(\frac{\partial A}{\partial t_2} \sin \tau + A \frac{\partial \varphi}{\partial t_2} \cos \tau\right) - f\left(y_0, \frac{\partial y_0}{\partial \tau}\right)
\]

Solution. With the expansions as described in part (a), we see that the \( O(\varepsilon) \) term is:

\[
2 \frac{\partial^2}{\partial u \partial v} y_0 + \frac{\partial^2}{\partial u^2} y_1 + f\left(y_0, \frac{\partial y_0}{\partial u}\right) + y_1 = 0.
\]

Upon substituting \( y_0 = A(v) \cos(u + \varphi(v)) \), it is easy to check that,

\[
2 \frac{\partial^2}{\partial u \partial v} y_0 = -2\left(\frac{\partial A}{\partial v} \sin(u + \varphi(v)) + A(v) \frac{\partial \varphi}{\partial v} \cos(u + \varphi(v))\right).
\]

Moreover, note that under the change of variables \( u \rightarrow u + \varphi(v) \) the operator \( \partial / \partial (u + \varphi(v)) \) is equivalent to \( \partial / \partial u \). Thus, upon substitution, our \( O(\varepsilon) \) term agrees with the one prescribed. \[ \square \]

(c) In order to avoid secular terms, show that the conditions to be satisfied are:

\[
\frac{\partial A}{\partial t_2} = \frac{1}{2\pi} \int_0^{2\pi} f\left(y_0, \frac{\partial y_0}{\partial \tau}\right) \sin(\tau) d\tau
\]
\[ A \frac{\partial \phi}{\partial t_2} = \frac{1}{2\pi} \int_0^{2\pi} f \left( y_0, \frac{\partial y_0}{\partial \tau} \right) \cos(\tau) d\tau \]

**Solution.** To avoid secular terms we use the Fredholm alternative. That is, if \( y_1 \) is bounded and periodic in \( u + \phi(v) \) then, the right hand side of the equation must be orthogonal to the nullspace of the differential operator \( y_{uu} + y \). In other words, the right hand side must not contain terms proportional to \( \sin(u + \phi(v)) \) or \( \cos(u + \phi(v)) \). With \( f \) being smooth, there is no harm in assuming it has a Fourier representation of the form:

\[ f = \sum a(n) \sin(n\tau) + \sum b(n) \cos(n\tau). \]

where:

\[ a(n) = \frac{1}{2\pi} \int_0^{2\pi} f \left( y_0, \frac{\partial y_0}{\partial \tau} \right) \sin(n\tau) d\tau \quad \text{and} \quad b(n) = \frac{1}{2\pi} \int_0^{2\pi} f \left( y_0, \frac{\partial y_0}{\partial \tau} \right) \cos(n\tau) d\tau. \]

Using the result from part (b) we see that the coefficients of the non orthogonal terms of \( \sin(\tau) \) and \( \cos(\tau) \) are, respectively,

\[ 2 A \frac{\partial A}{\partial t_2} - \frac{1}{2\pi} \int_0^{2\pi} f \left( y_0, \frac{\partial y_0}{\partial \tau} \right) \sin(\tau) d\tau \]

and,

\[ 2 A \frac{\partial \phi}{\partial t_2} - \frac{1}{2\pi} \int_0^{2\pi} f \left( y_0, \frac{\partial y_0}{\partial \tau} \right) \cos(\tau) d\tau. \]

Since we want these to be equal to 0, the desired result follows. \( \square \)

(d) Use the above result to discuss the steady state and limit cycle when \( f = (y^4 - 1)y' \).

**Proof.** Using all of our previous results, our asymptotic expansion to satisfy:

\[ y_0 = A(v) \cos(u + \phi(v)) \]

where \( A(v) \) and \( \phi(v) \) satisfy:

\[ 2 \frac{\partial A}{\partial t_2} = \frac{1}{2\pi} \int_0^{2\pi} \left( A^4 \cos^4 \tau - 1 \right) (-A \sin \tau) A \sin(\tau) d\tau \]

and,

\[ 2 A \frac{\partial \phi}{\partial t_2} = \frac{1}{2\pi} \int_0^{2\pi} \left( A^4 \cos^4 \tau - 1 \right) (-A \sin \tau) A \cos(\tau) d\tau. \]

\( \square \)

We use software to compute the integrals, we arrive at the following equations:

\[ A'(v) = -\frac{A}{32} \left( A^4 - 8 \right) \]

\[ \phi'(v) = 0. \]
Thus,

\[ A(v) = \frac{4\sqrt{8} e^{v/4}}{\sqrt{e^{32C_1} + e^v}} \]

\[ \varphi(v) = C_2, \]

where \( C_1 \) and \( C_2 \) are constants to be determined by initial conditions. All together, \( y \) has the form:

\[ y \sim \frac{4\sqrt{8} e^{v/4}}{\sqrt{e^{32C_1} + e^v}} \cos(u + C_2). \]

Note that, in the limit, as \( v \) goes to infinity, the coefficient approaches \( \sqrt{8} \). Therefore, the solution approaches the orbit:

\[ y \sim \sqrt{8} \cos(u + C). \]

**H4-2 (from Holmes 3.15).** This problem considers the case where the logistic equation has a slowly varying carrying capacity \( c(\epsilon t) \). The equation is (Shepherd and Stojkov, 2007)

\[ y' = r y \left( 1 - \frac{y}{c(\epsilon t)} \right) \quad \text{for} \quad 0 < t, \]

where \( y(0) = \alpha \) and \( r \) and \( \alpha \) are positive constants. It is assumed that \( c(\tau) \) is a smooth positive function. Show that a first-term approximation that is valid for large \( t \) is

\[ y \sim \frac{c(\epsilon t)}{1 + A_0 c(\epsilon t) e^{-rt}}, \]

where \( A_0 = \frac{1}{\alpha} - \frac{1}{c(0)} \). Make sure to explain your reasoning in how you determine the dependence of \( y_0 \) on the slow time variable \( t_2 = \epsilon t \).

**Solution.** Let \( u = t \) and \( v = \epsilon t \). Moreover, assume that \( y \) has an asymptotic expansion of the form:

\[ y(u, v) = y_0(u, v) + \epsilon y_1(u, v) + \cdots. \]

Furthermore, note that:

\[ \frac{d}{dt} = \frac{\partial}{\partial u} + \epsilon \frac{\partial}{\partial v}. \]

Upon substitution we get the \( O(1) \) to be:

\[ \frac{\partial}{\partial u} y_0(u, v) = r y_0(u, v) \left( 1 - \frac{y_0(u, v)}{c(v)} \right). \]

To avoid clutter we drop the notation of the arguments of \( y_0 \). The above can be rewritten as:

\[ \left( \frac{1}{y_0} + \frac{1}{c(v)} \left( 1 - \frac{y_0}{c(v)} \right) \right) \frac{\partial y_0}{\partial u} = r. \]
Thus, integrating against the $u$ variable leads to:

$$\ln |y_0| - \ln \left| 1 - \frac{y_0}{c} \right| = ru + f_0,$$

where $f_0$ is a function of $v$. Equivalently,

$$\frac{y_0}{1 - \frac{y_0}{c}} = A_0 e^{ru}.$$

Where simple algebraic manipulations lead to:

$$y_0 = \frac{c(v)}{1 + A_0 e^{-ru}}.$$

Without loss of generality we let $1/A_0$ be a function of $v$ denoted by $A_0$. Thus, we arrive at the equation prescribed:

$$y_0 = \frac{c(v)}{1 + A_0 c(v) e^{-ru}}.$$

It is left to find the equation corresponding to $A_0(v)$. To that end, note that $y(0) = \alpha$ implies:

$$\alpha = \frac{c(0)}{1 + A_0 c(0)}.$$

Simple manipulations yield the desired result $A_0 = \alpha^{-1} - c(0)^{-1}$. \hfill \Box

**H4-3 (from Holmes 3.35).** Consider the problem

$$\frac{d}{dt} \left( D(\varepsilon t) \frac{dy}{dt} \right) + y = 0 \quad \text{for } 0 < t,$$

where $y(0) = \alpha$ and $y'(0) = \beta$. The coefficient $D(\tau)$ is a smooth positive function with $D' > 0$. Find a first-term approximation of the solution valid for large $t$.

**Solution.** Note that this problem is akin of a slowly varying coefficient. Thus, we let $v = \varepsilon t$ be our slow time variable, and we will let $u = f(\varepsilon, t)$ be our fast time variable to be determined. Note that the problem is equivalent to:

$$D(v) \frac{d^2 y}{dt^2} + \varepsilon D'(v) \frac{dy}{dt} + y = 0.$$

Moreover, with the change of variables described above, we have that:

$$\frac{d}{dt} = \frac{\partial f}{\partial t} \frac{\partial}{\partial u} + \varepsilon \frac{\partial}{\partial v},$$

$$\frac{d^2}{dt^2} = \left( \frac{\partial f}{\partial t} \right)^2 \frac{\partial^2}{\partial u^2} + \frac{\partial^2 f}{\partial t^2} \frac{\partial}{\partial u} + 2\varepsilon \frac{\partial f}{\partial t} \frac{\partial^2}{\partial u \partial v} + \varepsilon^2 \frac{\partial^2}{\partial v^2}.$$
Now we let $y \sim y_0(u, v) + \varepsilon y_1(u, v) + \cdots$. From now on we use subscript notation to represent partial derivatives. Substitution leads to the following $O(1)$ term:

$$D(\varepsilon t) \left( f_t^2 \frac{\partial^2}{\partial u^2} + f_{tt} \frac{\partial}{\partial u} \right) y_0 + y_0 = 0,$$

equivalently,

$$\left( f_t^2 \frac{\partial^2}{\partial u^2} + f_{tt} \frac{\partial}{\partial u} \right) y_0 + \frac{1}{D(\varepsilon t)} y_0 = 0.$$

Our hope is to balance the terms $f_t^2 \frac{\partial^2}{\partial u^2}$ and $\frac{1}{D}$. Thus, we set them equal to each other. It follows that:

$$u_t = f_t = \frac{1}{\sqrt{D(\varepsilon t)}},$$

$$u = f = \int D(\varepsilon s)^{-1/2} ds,$$

and

$$u_{tt} = f_{tt} = -\varepsilon \frac{D'(\varepsilon t)}{2(D(\varepsilon t))^{3/2}}.$$

With $f$ at our hand we see that the time derivative operators have the form:

$$\frac{d}{dt} = u_t \frac{\partial}{\partial u} + \varepsilon \frac{\partial}{\partial v},$$

$$\frac{d^2}{dt^2} = (u_t)^2 \frac{\partial^2}{\partial u^2} + u_{tt} \frac{\partial}{\partial u} + 2\varepsilon u_t \frac{\partial^2}{\partial u \partial v} + \varepsilon^2 \frac{\partial^2}{\partial v^2}.$$
to:
\[
\left( \frac{\partial^2}{\partial u^2} + 1 \right) (y_1) + \left( \frac{D'(\varepsilon t)}{2\sqrt{D(\varepsilon t)}} A(v) + 2\sqrt{D(\varepsilon t)} A'(v) \right) \cos(u) \\
- \left( \frac{D'(\varepsilon t)}{2\sqrt{D(\varepsilon t)}} B(v) + 2\sqrt{D(\varepsilon t)} B'(v) \right) \sin(u) = 0.
\]

Invoking the Fredholm alternative, the operator $\partial_{uu} + 1$ does not contain secular terms if and only if the sine and cosine coefficients are identically equal to zero. Note that both coefficients lead to the same ODE, namely:
\[
\frac{D'(\varepsilon t)}{2\sqrt{D(\varepsilon t)}} A(v) + 2\sqrt{D(\varepsilon t)} A'(v) = 0.
\]

The above can be solved by the method of integrating factors, via the constraint of $D' > 0$. Thus, we have:
\[
A(v) = C_1 D(\varepsilon t)^{-1/4}
\]
and likewise,
\[
B(v) = C_2 D(\varepsilon t)^{-1/4}.
\]

Thus,
\[
y_0 = D(\varepsilon t)^{-1/4} [C_1 \sin(u) + C_2 \cos(u)].
\]

With the prescribed initial conditions we see that the following hold:
\[
\alpha = D(0)^{-1/4} C_2,
\]
and,
\[
\beta = D(0)^{-3/4} C_1 + O(\varepsilon).
\]

We conclude that, at leading order, $y$ has the following asymptotic expansion,
\[
y \sim D(\varepsilon t)^{-1/4} [\beta D(0)^{3/4} \sin(u) + \alpha D(0)^{1/4} \cos(u)],
\]
with,
\[
u = \int_0^t D(\varepsilon s)^{-1/2} ds.
\]
\]
\]