Notes on Perturbation methods

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Abstract

Notes relevant to perturbation methods. The following is not intended to be a rigorous set of notes, rather, a summary and collection of bullet points that summarizes the ideas discussed at the course M8402, as taught by Dr. Mori, Yoichiro.

1 Multiple scale Method

Consider the equation:

\[ y'' + \varepsilon y' + y = 0 \quad (1.1) \]

where \( y(0) = 0 \) and \( y'(0) = 1 \).

The “standard” power series expansion, namely:

\[
y(t) \sim y_0(t) + \varepsilon y_1(t) + \cdots \]

yields the solution:

\[
y(t) \sim \sin(t) - \frac{1}{2} \varepsilon t \sin(t). \]

The problem is that this solution grows unbounded for large \( t \). Hence, we should not expect that it agrees with the exact solution. In fact, the analytical solution is:

\[
y(t) = \frac{1}{\sqrt{1-\varepsilon^2/4}} e^{-\varepsilon t/2} \sin \left( t \sqrt{1-\varepsilon^2/4} \right). \]

Without much motivation, we assume that our system depends on at least two time scales, \( u = t \) and \( v = \varepsilon t \). Then, for the function \( y(t) = y(u, v) \) we have

\[
\frac{d}{dt} \mapsto \frac{\partial}{\partial u} \frac{du}{dt} + \frac{\partial}{\partial v} \frac{dv}{dt} = \frac{\partial}{\partial u} + \varepsilon^\alpha \frac{\partial}{\partial v},
\]

and analogously,

\[
\frac{d^2}{dt^2} \mapsto \frac{\partial^2}{\partial u^2} + 2 \varepsilon^\alpha \frac{\partial^2}{\partial u \partial v} + \varepsilon^{2\alpha} \frac{\partial^2}{\partial v^2}.
\]

With the above construction, we assume that \( y(u,v) \) has asymptotic expansion:

\[
y(u, v) \sim y_0(u,v) + \varepsilon y_1(u,v) + \varepsilon^2 y_2(u,v) + \cdots. \quad (1.2)\]
To avoid clutter, we omit the argument notation in what follows. Note that substitution of (1.2) into (1.1) yields:

\[ \left( \frac{\partial^2}{\partial u^2} + 2\varepsilon \frac{\partial^2}{\partial u \partial v} + \varepsilon^2 \frac{\partial^2}{\partial v^2} \right) (y_0 + \varepsilon y_1 + \cdots) + \varepsilon \left( \frac{\partial}{\partial u} + \varepsilon \frac{\partial}{\partial v} \right) (y_0 + \varepsilon y_1 + \cdots) + y_0 + \varepsilon y_1 + \cdots = 0 \]

We balance the equations with \( \alpha = 1 \) as to avoid terms that grow unbounded for large values of \( t \). Now, we proceed to collect terms in terms of \( \varepsilon \). We get the following:

- **\( O(1) \):** \( \left( \frac{\partial^2}{\partial u^2} + 1 \right) y_0 = 0 \), with \( y_0(0,0) = 0 \) and \( \frac{\partial}{\partial u} y_0(0,0) = 1 \).
  
  The general solution of the above system is:
  
  \[ y_0 = A \sin(u) + B \cos(u) \]
  
  where \( A \) and \( B \) are both functions of \( v \) such that \( A(0) = 1 \) and \( B(0) = 0 \).

- **\( O(\varepsilon) \):** \( \left( \frac{\partial^2}{\partial u^2} + 1 \right) y_1 = -2 \frac{\partial}{\partial u \partial v} y_0 - \frac{\partial}{\partial u} y_0 \), with \( y_1(u,v) = 0 \), and \( \frac{\partial}{\partial v} y_1 = -\frac{\partial}{\partial u} y_0 \), when \((u,v) = (0,0) \).

  Using the solution found for \( y_0 \) above we find the solution for \( y_1 \); namely:

  \[ y_1 = C \sin(u) + D \cos(u) - \frac{1}{2} (2B' + B) u \cos(u) - \frac{1}{2} (2A' + A) u \sin(u) \]

  where \( C \) and \( D \) are functions of \( v \), \( C(0) = A'(0) \), and \( D(0) = 0 \).

  Note that the solution of \( y_1 \) contains unbounded terms; namely, those that have a factor of \( u \). However, here we can avoid them by letting their respective coefficients be identically 0. In other words, we let:

  \[ 2B' + B = 0 \]

  and

  \[ 2A' + A = 0. \]

  Then, using some of the initial value conditions, we have that:

  \[ A = e^{v/2}, \]

  and

  \[ B = 0. \]

  All put together, we have our first term approximation:

  \[ y \sim e^{t/2} \sin(t). \]
2 Slowly Varying Coefficients

Consider the problem
\[ \frac{d}{dt} \left( D(\varepsilon t) \frac{dy}{dt} \right) + y = 0 \quad \text{for} \quad 0 < t, \]
where \( y(0) = \alpha \) and \( y'(0) = \beta \). The coefficient \( D(t) \) is a smooth positive function with \( D' > 0 \).

We compute the leading order approximation of the solution as follows.

Let \( v = \varepsilon t \) be our slow time variable, and we will let \( u = f(\varepsilon, t) \) be our fast time variable to be determined.

Note that the problem is equivalent to:
\[ D \left( f^2 \frac{\partial^2}{\partial u^2} + f_{tt} \frac{\partial}{\partial u} \right) y_0 + 1 = 0. \]

Moreover, with the change of variables described above, we have that:
\[ \frac{d}{dt} = \frac{df}{dt} \frac{\partial}{\partial u} + \varepsilon \frac{\partial}{\partial v} \]
\[ \frac{d^2}{dt^2} = \left( \frac{df}{dt} \right)^2 \frac{\partial^2}{\partial u^2} + \left( \frac{df}{dt} \right) \frac{\partial}{\partial u} + 2 \varepsilon \frac{df}{dt} \frac{\partial^2}{\partial u \partial v} + \varepsilon^2 \frac{\partial^2}{\partial v^2}. \]

Now we let \( y \sim y_0(u, v) + \varepsilon y_1(u, v) + \cdots \). From now on we use subscript notation to represent partial derivatives. Substitution leads to the following \( O(1) \) term:
\[ D(\varepsilon t) \left( f_{tt} \frac{\partial^2}{\partial u^2} + f_{tt} \frac{\partial}{\partial u} \right) y_0 + y_0 = 0, \]
equivalently,
\[ \left( f_{tt} \frac{\partial^2}{\partial u^2} + f_{tt} \frac{\partial}{\partial u} \right) y_0 + \frac{1}{D(\varepsilon t)} y_0 = 0. \]

Our hope is to balance the terms \( f_{tt} \frac{\partial^2}{\partial u^2} \) and \( \frac{1}{D} \). Thus, we set them equal to each other. It follows that:
\[ u_t = f_t = \frac{1}{\sqrt{D(\varepsilon t)}}, \]
\[ u = f = \int^t D(\varepsilon s)^{-1/2} ds, \]
and
\[ u_{tt} = f_{tt} = -\varepsilon \frac{D'(\varepsilon t)}{2 (D(\varepsilon t))^{3/2}}. \]

With \( f \) at our hand we see that the time derivative operators have the form:
\[ \frac{d}{dt} = u_t \frac{\partial}{\partial u} + \varepsilon \frac{\partial}{\partial v} \]
\[ \frac{d^2}{dt^2} = (u_t)^2 \frac{\partial^2}{\partial u^2} + u_{tt} \frac{\partial}{\partial u} + 2 \varepsilon u_t \frac{\partial^2}{\partial u \partial v} + \varepsilon^2 \frac{\partial^2}{\partial v^2}. \]

With \( y \sim y_0(u, v) + \varepsilon y_1(u, v) + \cdots \), the terms from our original equation are of the form:
\[ D \frac{d^2 y}{dt^2} = \left( \frac{\partial^2}{\partial u^2} y_0 \right) + \varepsilon \left( \frac{\partial^2}{\partial u^2} y_1 + 2 \sqrt{D} \frac{\partial^2}{\partial u \partial v} y_0 - \frac{D'}{2 \sqrt{D}} \frac{\partial}{\partial u} y_0 \right) + O(\varepsilon^2) \]
\[ \varepsilon D \frac{dy}{dt} = \varepsilon \left( \frac{D'}{D} \frac{\partial}{\partial u} y_0 + D' \frac{\partial}{\partial v} y_0 \right) + O(\varepsilon^2) \]

We proceed to inspect the term of our initial problem term-wise. We obtain the following:

\[ D(\varepsilon)^2 \frac{d^2}{dt^2} (y) = D(\varepsilon t) \left( \frac{1}{\sqrt{D(\varepsilon t)}} \frac{\partial^2}{\partial u^2} y_0 + \frac{D'(\varepsilon)}{2\sqrt{D(\varepsilon t)}} \frac{\partial}{\partial u} + 2\varepsilon \sqrt{D(\varepsilon t)} \frac{\partial^2}{\partial u \partial v} + \varepsilon^2 D(\varepsilon t) \frac{\partial^2}{\partial v^2} \right) (y) \]

\[ \varepsilon D'(\varepsilon t) \frac{d}{dt} (y) = \varepsilon D'(\varepsilon t) \left( \frac{1}{\sqrt{D(\varepsilon t)}} \frac{\partial}{\partial u} + \varepsilon \frac{\partial}{\partial v} \right) (y) \]

Now, reusing our assumption of \( y \) having asymptotic expansion \( y_0(u, v) + \varepsilon y_1(u, v) + \cdots \), we obtain the 1 and \( \varepsilon \) order terms:

\[ O(1) : \left( \frac{\partial^2}{\partial u^2} + 1 \right) (y_0) = 0 \]

\[ O(\varepsilon) : \left( \frac{\partial^2}{\partial u^2} + 1 \right) (y_1) + \left( \frac{D'(\varepsilon t)}{2\sqrt{D(\varepsilon t)}} \frac{\partial}{\partial u} + 2\varepsilon \sqrt{D(\varepsilon t)} \frac{\partial^2}{\partial u \partial v} \right) (y_0) = 0. \]

It should be clear that the \( O(1) \) term has a solution of the form

\[ y_0(u, v) = A(v) \sin(u) + B(v) \cos(u), \]

with \( A(v) \) and \( B(v) \) are functions to be determined. Substituting \( y_0 \) into the \( O(\varepsilon) \) term leads to:

\[ \left( \frac{\partial^2}{\partial u^2} + 1 \right) (y_1) + \left( \frac{D'(\varepsilon t)}{2\sqrt{D(\varepsilon t)}} A(v) + 2\varepsilon \sqrt{D(\varepsilon t)} A'(v) \right) \cos(u) - \left( \frac{D'(\varepsilon t)}{2\sqrt{D(\varepsilon t)}} B(v) + 2\varepsilon \sqrt{D(\varepsilon t)} B'(v) \right) \sin(u) = 0. \]

Invoking the Fredholm alternative, the operator \( \partial_{uu} + 1 \) does not contain secular terms if and only if the sine and cosine coefficients are identically equal to zero. Note that both coefficients lead to the same ODE, namely:

\[ \frac{D'(\varepsilon t)}{2\sqrt{D(\varepsilon t)}} A(v) + 2\varepsilon \sqrt{D(\varepsilon t)} A'(v) = 0. \]

The above can be solved by the method of integrating factors, via the constraint of \( D' > 0 \). Thus, we have:

\[ A(v) = C_1 D(\varepsilon t)^{-1/4} \]

and likewise,

\[ B(v) = C_2 D(\varepsilon t)^{-1/4}. \]

It follows that,

\[ y_0 = D(\varepsilon t)^{-1/4}[C_1 \sin(u) + C_2 \cos(u)]. \]

With the prescribed initial conditions we see that the following hold:

\[ \alpha = D(0)^{-1/4} C_2, \]

and,

\[ \beta = D(0)^{-3/4} C_1 + O(\varepsilon). \]
We conclude that, at leading order, \( y \) has the following asymptotic expansion,
\[
y \sim D(\varepsilon t)^{-1/4}[\beta D(0)^{3/4} \sin(u) + \alpha D(0)^{1/4} \cos(u)],
\]
with,
\[
u = \int^t D(\varepsilon s)^{-1/2} ds.
\]

3 The Method of Averaging (Krylov-Bogoliubov).

Consider the following equation:
\[
x'' \mu f(x, x') + p^2 x = 0,
\]
where \( \mu \) and \( p \) are constant and the prime denoted differentiation with respect to \( t \). Without elaborating on the intermediate steps, the approximate solution is given by
\[
x = A(t) \sin (pt + \varphi(t))
\]
and
\[
x' = pA(t) \cos (pt + \varphi(t))
\]
where \( A(t) \) and \( \varphi(t) \) are determined by the equations:
\[
A' = -\frac{\mu}{2\pi p} \int_0^{2\pi} f(A \sin \psi, Ap \cos \psi) \sin \psi d\psi
\]
and
\[
\varphi' = \frac{\mu}{2\pi p} \int_0^{2\pi} f(A \sin \psi, Ap \cos \psi) \sin \psi d\psi.
\]
Note that the approximation is proportional to \( |\mu f| \). We motivate the previous result with the following example.

Consider the equation:
\[
x'' + \varepsilon(t)x' + \eta(t)x = 0,
\]
which may be written as:
\[
x'' + [(\eta - p^2)x + \varepsilon x'] + p^2 x = 0.
\]
Note that the latter is equivalent to \( (3.1) \) with \( \mu = 1 \) and \( f(x, x') = (\eta - p^2)x + \varepsilon x' \). Using equations (3.4) and (3.5) \( A(t) \) and \( \varphi(t) \) satisfy:
\[
A = C_1 \exp \left( -\frac{1}{2} \int \varepsilon dt \right)
\]
and
\[
\varphi = \frac{1}{2p} \int \eta dt - \frac{pt}{2} + C_2,
\]
where \( C_1 \) and \( C_2 \) are integration constants. Hence, the approximate solution of \( (3.2) \) and \( (3.3) \) are:
\[
x = C_1 \exp \left( -\frac{1}{2} \int \varepsilon dt \right) \sin \left( \frac{pt}{2} + \frac{1}{2} \int \eta dt + C_2 \right)
\]
and
\[
x' = C_1 p \exp \left( -\frac{1}{2} \int \varepsilon dt \right) \cos \left( \frac{pt}{2} + \frac{1}{2} \int \eta dt + C_2 \right).
\]
4 The WKB method

Consider the equation:

\[ \varepsilon^2 y'' - q(x)y = 0. \] (4.1)

If we assume \( q(x) \) to be constant then, the general solution of the above problem is:

\[ y(x) = a_0 e^{-\sqrt{q}/\varepsilon} + b_0 e^{\sqrt{q}/\varepsilon}. \]

It seems a natural choice to assume that solution of this type of equations have an asymptotic expansion that involves exponential terms. Without being rigorous, the asymptotic expansion of a solution has the form:

\[ y \sim e^{\theta(x)/\sqrt{\varepsilon}} [y_0(x) + \varepsilon^\alpha y_1(x) + \cdots]. \] (4.2)

To avoid clutter we drop the argument notation. Note that

\[ y' \sim e^{\theta/\sqrt{\varepsilon}} (e^{-\alpha} \theta' y_0 + y'_0 + \theta' y_1 + \cdots), \]

and

\[ y'' \sim e^{\theta/\sqrt{\varepsilon}} [e^{-2\alpha} \theta_x y_0 + \varepsilon^{-\alpha} (\theta_{xx} y_0 + 2 \theta_x y'_0 + \theta_x^2 y_1) + \cdots] \]

We proceed by substituting (4.2) into (4.1). We obtain:

\[ \varepsilon^2 \left( \frac{1}{\varepsilon^{2\alpha}} (\theta_x)^2 y_0 + \frac{1}{\varepsilon^\alpha} [\theta'' y_0 + 2 \theta' y'_0 + (\theta_x)^2 y_1] + \cdots \right) - q(x) (y_0 + \varepsilon^\alpha y_1 + \cdots) = 0 \] (4.3)

Moreover, if we balance the \( \varepsilon \) terms with \( \alpha = 1 \) we obtain the following:

- \( O(1) \) : \( (\theta_x)^2 = q(x) \). This term is called the \textit{eikonal equation} and has solution:
  \[ \theta(x) = \pm \int \sqrt{q(x)} \, dx \]

- \( O(\varepsilon) \) : \( \theta'' y_0 + 2 \theta' y'_0 + (\theta_x)^2 y_1 = q(x) y_1 \). This is the \textit{transport equation}. Note that \( \theta(x) \) satisfies the eikonal equation thus, we can simplify the above to \( \theta'' y_0 + 2 \theta' y'_0 = 0 \), with solution:
  \[ y_0(x) = \frac{c}{\sqrt{\theta_x}}, \]

where \( c \) is an arbitrary constant.

Thus, the first term solution to (4.1) is:

\[ y(x) \sim q(x)^{1/4} \left[ a_0 \exp \left( -\frac{1}{\varepsilon} \int \sqrt{q(x)} \, dx \right) + b_0 \exp \left( \frac{1}{\varepsilon} \int \sqrt{q(x)} \, dx \right) \right] \]

5 References