Note. The solutions exhibited here are greatly influenced by Paul Garret’s online notes: http://www-users.math.umn.edu/~garrett/ and by more than fruitful discussions with my colleague Hailee Peck.

Problem [1]. Given \( \varepsilon > 0 \), construct an open set \( U \subset \mathbb{R} \) containing \( \mathbb{Q} \) and with Lebesgue measure less than \( \varepsilon \).

Proof. Let \( q_i \) be an enumeration of the rationals where \( i \in \mathbb{N} \). Then, cover \( q_i \) by an open ball \( U_i \) of measure \( x_i \) with \( x_i \in (0, 1) \). By the sub additive property of the Lebesgue measure \( \mu \),

\[
\mu(U) = \mu\left(\bigcup U_i\right) \\
\leq \sum \mu(U_i) \\
= \frac{x}{1-x}
\]

It then suffices to chose \( x < \varepsilon/(1+\varepsilon) \).

Problem [2]. Give an example of a sequence \( \{f_n\} \) of continuous functions on \([0,1]\) such that \( \lim_{n} f_n(x) = 0 \) for all \( x \in [0,1] \) but \( \int_{0}^{1} f_n(t) \, dt = 1 \) for all \( n \).

Solution. This can be achieved by tent functions. Let \( f_n(x) \) be the function that linearly interpolates the points \((0,0),(1/n,n),(2/n,0)\) and it’s equal to 0 everywhere else. Clearly \( \int_{0}^{1} f_n(t) \, dt = 1 \). Furthermore, for all \( x > 0 \) there exist \( n \) such that \( 2/n < x \) and thus, \( f_n(x) \to 0 \) pointwise.

Problem [3]. Suppose \( f \in L^1(\mathbb{R}) \) and \( \int_{a}^{b} f(x) \, dx = 0 \) for all \( a,b \in \mathbb{R} \). Show that \( f(x) = 0 \) almost everywhere.

Proof. We proceed by contradiction, assuming \( f \) is positive (or negative, followed by a similar argument) on a set \( E \subset [c,d] \) of positive measure (for some \( c,d \in \mathbb{R} \)). Let \( F \) be a closed subset of \( E \) that also has positive measure. Since \( f > 0 \) on \( E \), we have \( \int_{F} f \, dx > 0 \) (note that the inequality is strict).

Consider the set \( U = [c,d] - F \). It is clearly open, so it can be written as a disjoint union of open intervals; i.e.

\[
U = \bigcup_{k=1}^{\infty} (a_k,b_k).
\]

Now, since \( f \) is measurable we have

\[
0 = \int_{c}^{d} f = \int_{U} f + \int_{F} f.
\]
Recall that the last integral above is strictly positive and thus, \( \int_U f \) is strictly negative. This is a contradiction as

\[
\int_U f = \sum_{k=1}^{\infty} \int_{a_k}^{b_k} f = \sum_{k=1}^{\infty} 0 = 0.
\]

The last part is fully justified by Fubini-Tonelli theorem as \( f \in L^1(\mathbb{R}) \).

**Problem [4].** Let \( f \in L^2([0,1]) \) be differentiable almost everywhere, with derivative \( f' \in L^2([0,1]) \). Show that there is a constant \( C \) such that \( |f(x) - f(y)| < C \cdot |x - y|^{1/2} \) for \( x, y \in [0,1] \).

**Proof.** Observe that \( |f|_{L^2} + |f'|_{L^2} \) is finite and thus, \( f \) is in the space \( H^1[0,1] \) (the Sobolev space \( W^{1,2}[0,1] \)). Recall that in this space smooth functions are dense; i.e. there exist a sequence \( \{f_n\} \) of functions in \( H^1[0,1] \cap C^\infty[0,1] \) such that \( f_n \to f \) in \( H^1[0,1] \). Then, it suffices to prove the statement for \( f \in C^\infty[0,1] \). This will follow from the fundamental theorem of calculus and Hölder’s inequality:

\[
f(x) - f(y) = \int_y^x f'(s) \, ds
\]

then,

\[
|f(x) - f(y)| \leq \int_y^x |f'(s)| |1| \, ds
\]

then, by Hölder’s

\[
\leq |f'(x)|_{L^2[x,y]} \left( \int_y^x 1 \, ds \right)^{1/2}
\]

\[
\leq |f'|_{L^2[0,1]} \cdot |x - y|^{1/2}
\]

as wanted.

**Problem [5].** Let \( C = \{ v = (v_1, v_2, \ldots) \in \ell^2 : |v_n| \leq \frac{1}{n^2} \} \subset \ell^2 \). Show that \( C \) is compact.

**Proof.** The exposition from Paul Garret at Discussion 2, example 5 is excellent.

**Problem [6].** Let \( E \) be a Lebesgue measurable subset of \([0,1]\) and let \( f(x) = \int_E \sin(tx) \, dt \).

Show that \( f(x) \) is continuous.

**Proof.** Note that

\[
\int_E \sin(tx) \, dt = \int_{\mathbb{R}} \chi_E \sin(tx) \, dt
\]

where \( \chi_E \) is the characteristic function of the set \( E \). Riemann Lebesgue lemma asserts that the Fourier transform of a function in \( L^1 \) is continuous, which is the case here.

**Problem [7].** For \( 1 < p < \infty, f \in L^p(\mathbb{R}), \) and \( g \in L^1(\mathbb{R}), \) show that \( |f * g|_{L^p} \leq |f|_{L^p} \cdot |g|_{L^1} \).

**Proof.** This is an instance of Young’s inequality for convolutions. A detailed proof can be found here. However, a more general result is discussed in Paul Garret’s Real Analysis notes, (Discussion 12, Problem 3).

**Theorem (Young’s Inequality for convolutions.).** Let \( p, q, r \in \mathbb{R}_{\geq 1} \) satisfy

\[
\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.
\]

Let \( f \in L^p(\mathbb{R}^n) \) and \( g \in L^q(\mathbb{R}^n) \). Then \( f * g \in L^r(\mathbb{R}^n) \) with

\[
|f * g|_{L^r} \leq |f|_{L^p} \cdot |g|_{L^q}.
\]
The desired result follows by letting $q = 1$ and $r = p$ above.

\begin{proof}

\textbf{Problem [8].} Show that $C^1[a,b]$ with norm $|f|_{C^1} = \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |f'(x)|$ is a Banach space.

\end{proof}

Proof. Take for granted that this is a normed vector space (easy to check and it is not the point here). It is left to show that this space is complete. Thus, let $\{f_n\}$ be a Cauchy sequence of $C^1[a,b]$ under the proposed norm. Then, there exist $N$ such that

$$\sup_{x \in [a,b]} |f_n - f_m| + \sup_{x \in [a,b]} |f'_n - f'_m| < \varepsilon$$

for $n, m > N$. Recall that $C^0[a,b]$ is complete when equipped with the sup norm. Thus, $f_n \to f$ and $f'_n \to g$ uniformly in $C^0[a,b]$. From here, it is enough to prove that $f$ is differentiable and has derivative $g$. So, with this idea in mind, and by means of the fundamental theorem of calculus we set

$$f_n(x) - f_n(a) = \int_a^x f'_n(x).$$

The limit of the left hand side converges to $f(x) - f(a)$ and we can show that the limit on the right converges to $\int_a^x g$. Observe that, for $n$ sufficiently large

$$\left| \int_a^x f'_n(x) - \int_a^x g(x) \right| \leq \sup_{x \in [a,b]} \int_a^x |f'_n(x) - g(x)| \leq \varepsilon |b - a|.$$

Hence, $f(x) - f(a) = \int_a^x g$ and then, the Fundamental Theorem of Calculus asserts that $f' = g$ as we wanted. \qed