Selected exercises from *Abstract Algebra* by *Dummit and Foote* (3rd edition).

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Abril 12, 2017

Section 10.1

Exercise 8. An element $m$ of the $R$-module $M$ is called a torsion element if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$\text{Tor}(M) = \{ m \in M \mid rm = 0 \text{ for some non zero } r \in R \}$

(a) Prove that if $R$ is an integral domain then $\text{Tor}(M)$ is a submodule of $M$.

Proof. We use the submodule criterion. Observe that since $R$ is an integral domain $1 \neq 0$ and $1 \cdot 0 = 0$. Hence $\text{Tor}(M)$ is non empty. Now let $x, y$ be in $\text{Tor}(M)$ so that $r_1x = 0$ and $r_2y = 0$; and let $r$ be an arbitrary element of $R$. Then (using the fact that $R$ is commutative), observe that

$$r_1r_2(x + ry) = r_2(r_1x) + r_1r(r_2y) = 0 + 0 = 0.$$ 

Hence $x + ry$ is in the torsion as we wanted.

(b) Give an example of a ring $R$ and an $R$-module $M$ such that $\text{Tor}(M)$ is not a submodule.

Solution. Taking a hint from part (a) we want $R$ to not be an integral domain. Let $R = \mathbb{Z}/10\mathbb{Z}$ and consider the $R$-module generated by itself. Note that 2 and 5 are torsion elements as $2 \cdot 5 \equiv 0 \mod 10$, but $7 = 2 + 5$ is not, since $r \cdot 7 = 0$ has only the trivial solution. Hence $\text{Tor}(M)$ is not closed under addition, and therefore is not a submodule.

(c) If $R$ has zero divisors show that every nonzero $R$-module has non zero torsion elements.

Proof. Let $M$ be a nonzero $R$-module. Let $a$ and $b$ be the zero divisor elements in $R$ and take $m \neq 0 \in M$ ($m$ exists since $M$ is non zero). Then, if $b \cdot m = 0$ we are done, and $m$ is in the torsion. Otherwise $a(b \cdot m) = (ab) \cdot m = (0) \cdot m = 0$, and $(b \cdot m)$ is in the torsion as desired.

Exercise 12. In the notation of the preceding exercises prove the following facts about annihilators.

(a) Let $N$ be the submodule of $M$ and let $I$ be its annihilator in $R$. Prove that the annihilator of $I$ contains $N$. Give an example where the annihilator of $I$ in $M$ does not equal $N$.
Proof. Let \( n \in N \) and consider the product \( n \cdot i \) where \( i \) is an arbitrary element in \( I \). Since \( i \) is an element of the annihilator of \( N \) we see that \( n \cdot i = 0 \) as needed. 

Let \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) be a \( \mathbb{Z} \)-module and let \( \mathbb{Z}/2\mathbb{Z} \times 0 \) be a submodule. Let \( I = \text{Ann}(\mathbb{Z}/2\mathbb{Z} \times 0) = 2\mathbb{Z} \) and note that \( \text{Ann}(I) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \neq \mathbb{Z}/2\mathbb{Z} \times 0 \).

(b) Let \( I \) be a right ideal of \( R \) and let \( N \) be its annihilator in \( M \). Prove that the annihilator of \( N \) in \( R \) contains \( I \). Give an example where the annihilator of \( N \) in \( R \) does not equal \( I \).

Proof. Let \( i \) be an element of \( I \), and consider the product \( n \cdot i \) where \( n \) is an arbitrary element of \( N \). Since \( n \) is in the annihilator of \( I \) in \( M \). Then \( n \cdot i = 0 \) as required. 

Let \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \) be a \( \mathbb{Z} \)-module, and let \( i = 4\mathbb{Z} \) be a right ideal. Note that \( \text{Ann}(4\mathbb{Z}) \) (in \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \)) is the set 
\[
\{(a, b) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \mid 4na = 0 \mod 2, 4nb = 0 \mod 6 \text{ for all } n\}
\]

Thus, 
\[
\text{Ann}(4\mathbb{Z}) = \{(0, 0), (0, 3), (1, 0), (1, 3)\}.
\]

Then we inspect the annihilator in \( \mathbb{Z} \) of \( \{(0, 0), (0, 3), (1, 0), (1, 3)\} \). Note that in order to get 0 modulo 2 in the first coordinate, and element in the annihilator must be an even integer. Then, we are done as the second coordinate becomes 0 modulo 6. Hence 
\[
I = 4\mathbb{Z} \neq \text{Ann}(\text{Ann}(I)) = 2\mathbb{Z}.
\]

Exercise 18. Let \( F = \mathbb{R} \), let \( V = \mathbb{R}^2 \) and let \( T \) be the linear transformation from \( V \) to \( V \) which is rotation clockwise about the origin by \( \pi/2 \) radians. Show that \( V \) and \( 0 \) are the only \( F[x] \)-submodules for this \( T \).

Proof. Analogously, we can show that the only \( T \) stable subspaces of \( \mathbb{R}^2 \) are \( 0 \) and \( \mathbb{R}^2 \). Let \( N \) be a \( T \) stable subspace and let \( n \) be its dimension. Since we are in \( \mathbb{R}^2 \) it suffices to show that if \( n = 1 \), then the space is not stable. Just note that in \( \mathbb{R}^2 \) orthogonal vectors are linearly independant. Therefore, if \( v \) is the basis vector of a 1 dimensional space \( N \), \( T(v) \) is independant from \( v \) and hence is not in \( T(N) \). The conclusion then follows as \( 0 \) and \( \mathbb{R}^2 \) are the trivial invariant subspaces.

Exercise 20. Let \( F = \mathbb{R} \), let \( V = \mathbb{R}^2 \) and let \( T \) be the linear transformation from \( V \) to \( V \) which is rotation clockwise about the origin by \( \pi \) radians. Show that every subspace of \( V \) is an \( F[x] \)-submodule for this \( T \).

Proof. As in the previous exercise, it suffices to show that every subspace of \( \mathbb{R}^2 \) is \( T \) invariant. Since \( 0 \) and \( \mathbb{R}^2 \) are trivially invariant we only need to show that any 1-dimensional space is \( T \) invariant. Let \( \{v\} \) be the basis of the proposed subspace (note \( v \neq 0 \)). Then any vector \( r \cdot v \) in the subspace satisfies 
\[
T(r \cdot v) = r \cdot T(v) = r \cdot (-v) = (-r) \cdot v.
\]

It is clear that \( T(v) \) is in the same subspace, as we wanted.
Section 10.2

Exercise 5. Exhibit all \( \mathbb{Z} \)-module homomorphisms from \( \mathbb{Z}/30\mathbb{Z} \) to \( \mathbb{Z}/21\mathbb{Z} \).

Exhibition. Note that \( \mathbb{Z}/30\mathbb{Z} \) is generated by 1, and furthermore, over the integers we have
\[
\varphi(rx + y) = rx\varphi(1) + y\varphi(1).
\]
Therefore, it suffices to define the map \( 1 \mapsto \varphi(1) \) to characterize the homomorphism. We proceed by looking at the order of 1. It is necessary that \( 0 = 30\varphi(1) \), therefore we look for solutions to
\[
30\varphi(1) \equiv 0 \mod 21.
\]
Since \( 21 = 3\times 7 \), \( \varphi(1) \) must be a multiple of 7. Then, the possible choices are 7, 14, and 21 and the homomorphisms are characterized by
\[
\varphi(1) = 7, \quad \varphi(1) = 14, \quad \varphi(1) = 21 = 0.
\]

Exercise 8. Let \( \varphi : M \to N \) be an \( R \)-module homomorphism. Prove that \( \varphi(\text{Tor}(M)) \subseteq \text{Tor}(N) \).

Proof. Let \( n \) be an arbitrary element of \( \varphi(\text{Tor}(M)) \). Then, there exists \( m \in \text{Tor}(M) \) such that \( n = \varphi(m) \). Furthermore, there exist \( r \in R \) such that \( rm = 0 \). Then, we see that
\[
0 = \varphi(0) = \varphi(rm) = r\varphi(m) = rn.
\]
Therefore \( n \in \text{Tor}(N) \) as desired. \( \square \)

Exercise 13. Let \( I \) be a nilpotent ideal in a commutative ring \( R \), let \( M \) and \( N \) be \( R \)-modules and let \( \varphi : M \to N \) be an \( R \)-module homomorphism. Show that if the induced map \( \bar{\varphi} : M/IM \to N/IN \) is surjective, then \( \varphi \) is surjective.

Proof. Observe that the map \( \bar{\varphi} \) is given by \( (m + IM) \mapsto (\phi(m) + IN) \). Then
\[
N/IN = \bar{\varphi}(M/IM) = (\varphi(M) + IN)/IN.
\]
Then, by the lattice isomorphism theorem
\[
N = \varphi(M) + IN.
\]
Before we proceed, we prove the following,
\[
I\varphi(M) \subseteq \varphi(M).
\]
Take \( n \in I\varphi(M) \) and note that \( n \) has the form \( i\varphi(m) \) for some \( i \in I \) and \( m \in M \). Then \( n = i\varphi(m) = \varphi(im) \), and it follows that \( n \in \varphi(M) \). Back to the equality \( N = \varphi(M) + IN \), we plugin recursively to see that
\[
N = \varphi(M) + IN
= \varphi(M) + I(\varphi(M) + IN)
= \varphi(M) + I\varphi(M) + I^2N
= \varphi(M) + I^2N \quad \text{since} \quad I\varphi(M) \subseteq \varphi(M).
\]
Inductively we see that \( N = \varphi(M) + I^n N \). In particular, since \( I \) is nilpotent, of say, order \( n \).

\[
N = \varphi(M) + I^n N = \varphi(M) + 0N = \varphi(M).
\]

Hence, \( \varphi \) is surjective as we wanted. \( \square \)

**Section 10.3**

**Exercise 5.** Let \( R \) be an integral domain. Prove that every finitely generated torsion \( R \)-module has a nonzero annihilator i.e., there is a nonzero element \( r \in R \) such that \( rm = 0 \) for all \( m \in M \) — here \( r \) does not depend on \( m \). Give an example of a torsion \( R \)-module whose annihilator is the zero ideal.

**Proof.** Let \( M \) be the torsion \( R \)-module. Then, any \( m \in M \) has the form

\[
m = r_1 g_1 + r_2 g_2 + \cdots + r_n g_n,
\]

where \( r_i \in R \), \( g_i \) is a generator, and \( n \) is finite as \( M \) is finitely generated. Let \( a_i \) be a nonzero element such that, \( a_i g_i = 0 \) (this element exist by definition of a torsion module) and consider the product \( p = \prod_{i=1}^{n} a_i \). Observe that \( p \) is non zero since \( R \) is an integral domain and \( a_i \neq 0 \) for all \( i \). Then, for an arbitrary element \( m \) of the torsion module \( M \) we have,

\[
pm = \sum_{i=1}^{n} \left( r_i a_i g_i \prod_{j \neq i} a_j \right) = 0.
\]

Thus, \( p \) is a non zero element in \( \text{Ann}_R M \), as desired. \( \square \)

**Exercise 9.** An \( R \)-module \( M \) is called irreducible if \( M \neq 0 \) and if \( 0 \) and \( M \) are the only submodules of \( M \). Show that \( M \) is irreducible if and only if \( M \neq 0 \) and \( M \) is a cyclic module with any nonzero element as generator. Determine all the irreducible \( \mathbb{Z} \)-modules.

**Proof.** First, assume \( M \) is irreducible. Then, by definition, \( M \) is not \( 0 \). Now, let \( m \) be any nonzero element in \( M \). Since \( 1 \in R \) then \( 1 \cdot m = m \) is in \( \text{Rin} \). The latter forces \( R \cdot m = M \). Hence \( M \) is cyclic and its generated by any nonzero element.

Now we assume that \( M \neq 0 \) and \( M \) is cyclic, generated by any non zero element. It suffices to show that \( 0 \) and \( M \) are the only submodules. Note that the additive identity of \( M \) generates the \( 0 \) submodule. On the other hand, if \( N \) is a nonzero submodule of \( M \) then \( N \) contains a nonzero element \( n \). Furthermore \( R \cdot n \subseteq N \) is a subset of \( M \). Since \( M \) is generated by any nonzero element we have \( R \cdot n = M \subseteq N \). Hence \( N = M \) as desired.

It follows that the irreducible \( \mathbb{Z} \)-modules are the cyclic groups of prime order. \( \square \)

**Exercise 10.** Assume \( R \) is commutative. Show that an \( R \)-module \( M \) is irreducible if and only if \( M \) is isomorphic (as an \( R \)-module) to \( R/I \) where \( I \) is a maximal ideal of \( R \).[By the previous exercise, if \( M \) is irreducible there is a natural map \( R \to M \) defined by \( r \mapsto rm \), where \( m \) is any fixed nonzero element of \( M \).]

**Proof.** First we assume that \( M \) is isomorphic to \( R/I \). Note that, since \( I \) is maximal, the quotient \( R/I \) is a field, with trivial ideals. Now, since a submodule of \( M \) is one to one correspondence with ideals in \( R/I \) (by the fourth isomorphism theorem), the only submodules of \( M \) are \( 0 \) and \( M \) itself. Hence \( M \) is irreducible.

Now we assume that \( M \) is irreducible. Taking the hint, we let \( \varphi : R \to M \) be given by \( \varphi(r) = rm \), for some fixed \( m \neq 0 \) in \( M \). By the first isomorphism theorem \( R/\ker(\varphi) \) is isomorphic to \( M \). It is left to show that \( \ker(\varphi) \) is a maximal ideal in \( R \). This is easy, as ideals
in $R/\ker(\varphi)$ are in one to one correspondence with submodules of $M$ (again, by the fourth isomorphism theorem). Since the only submodules of $M$ are 0 and $M$, it follows that the only ideals in $R/\ker(\varphi)$ are 0 and the entire quotient. Thus, $R/\ker(\varphi)$ is a field and $\ker(\varphi)$ is a maximal ideal as we wanted.

**Exercise 11.** Show that if $M_1$ and $M_2$ are irreducible $R$-modules, then any nonzero $R$-module homomorphism from $M_1$ to $M_2$ is an isomorphism. Deduce that if $M$ is irreducible then $\text{End}_R(M)$ is division ring. [Consider the kernel and the image.]

**Proof.** Let $\varphi$ be a nonzero homomorphism from $M_1$ to $M_2$. First we show that $\varphi$ is injective by proving that $\ker(\varphi) = \{0\}$. By way of contradiction, assume there exist $m_1 \neq 0$ in $\ker(\varphi)$. Then, since $M_1$ is irreducible and furthermore generated by $m_1$, for all $m \in M_1$ there exist $r \in R$ such that $rm_1 = m$. Then,

$$\varphi(m) = \varphi(rm_1) = r\varphi(m_1) = 0.$$  

It follows that $\varphi(m)$ is 0 for all $m \in M$, a contradiction since $\varphi$ is not a zero homomorphism. Now we show that $\varphi$ is surjective. Since $\varphi$ is not the zero homomorphism, there exist $m_2 \neq 0$ such that $m_2 = \varphi(m_1)$. Furthermore since $M_2$ is generated by $m_2$, we have that, for all $m \in M_2$, $m = rm_2$ for some $r \in R$. Thus,

$$m = rm_2 = r\varphi(m_1) = \varphi(rm_1).$$

Since $m$ is an arbitrary element in $M_2$ we have shown that $m$ is in the image of $\varphi$ for all $m \in M_2$. Hence, $\varphi$ being injective and surjective implies it is an isomorphism. Now consider the ring $\text{End}_R(M)$, where $M$ is irreducible. We have shown that any nonzero $\varphi$ in the ring is an isomorphism, hence invertible. Therefore, $\text{End}_R(M)$ satisfies the definition of a division ring. 

\[\square\]