Selected exercises from Abstract Algebra by Dummit and Foote (3rd edition).

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Section 11.1

Exercise 6. Let \( V \) be a vector space of finite dimension. If \( \varphi \) is any linear transformation from \( V \) to \( V \) prove that there is an integer \( m \) such that the intersection of the image of \( \varphi^m \) and the kernel of \( \varphi^m \) is \( \{0\} \).

Proof. Note that for all \( i \) we have

\[
\ker(\varphi^i) \subseteq \ker(\varphi^{i+1})
\]

and

\[
\operatorname{Im}(\varphi^{i+1}) \subseteq \operatorname{Im}(\varphi^i).
\]

Since \( V \) is finite dimensional, so are the kernel and the image therefore, there exist \( a, b \) such that \( \ker(\varphi^a) = \ker(\varphi^{a+1}) \) and \( \operatorname{Im}(\varphi^{b+1}) = \operatorname{Im}(\varphi^b) \). Without loss of generality let \( b \geq a \). We will show that \( \ker \varphi^b \cap \operatorname{Im} \varphi^b = \{0\} \). Any element \( v \in \ker \varphi^b \cap \operatorname{Im} \varphi^b \) satisfies:

i) \( v = \varphi^b(w) \) for some \( w \in V \), and

ii) \( 0 = \varphi^b(v) \).

By substitution, \( 0 = \varphi^b(v) = \varphi^b(\varphi^b(w)) = \varphi^{2b}(w) \). Therefore \( w \in \ker \varphi^{2b} \) and since \( 2b > b \), \( w \in \ker \varphi^b \) as well. Therefore

\[
v = \varphi^b(w) = 0
\]

as desired. \( \square \)

Exercise 7. Let \( \varphi \) be a linear transformation from a vector space \( V \) of dimension \( n \) that satisfies \( \varphi^2 = 0 \). Prove that the image of \( \varphi \) is contained in the kernel of \( \varphi \) and hence that the rank of \( \varphi \) is at most \( n/2 \).

Proof. We wish to show that \( \operatorname{Im}(\varphi) \subseteq \ker(\varphi) \).

Take \( v \in \operatorname{Im}(\varphi) \). Then, there exist \( w \) such that \( \varphi(w) = v \). Now, observe that

\[
\varphi(v) = \varphi(\varphi(w)) = \varphi^2(w) = 0
\]

Hence, \( v \) is in the kernel of \( \varphi \), and furthermore \( \operatorname{Im}(\varphi) \subseteq \ker(\varphi) \). It follows that

\[
n = \dim(V) = \dim(\operatorname{Im}(\varphi)) + \dim(\ker(\varphi)) \geq 2 \dim(\operatorname{Im}(\varphi))
\]

it is to say that \( \frac{n}{2} \geq \dim(\operatorname{Im}(\varphi)) \). \( \square \)
Exercise 8. Let $V$ be a vector space over $F$ and let $\varphi$ be a linear transformation of the vector space $V$ to itself. A non zero element $v \in V$ satisfying $\varphi(v) = \lambda v$ for some $\lambda \in F$ is called an eigenvector of $\varphi$ with eigenvalue $\lambda$. Prove that for any fixed $\lambda \in F$ the collection of eigenvectors of $\varphi$ with eigenvalue $\lambda$ together with $0$ forms a subspace of $V$.

Proof. We proceed by the standard method. Let $v_1$ and $v_2$ be eigenvectors of $\varphi$ with eigenvalue $\lambda$ and let $c_1$ and $c_2$ be elements in $F$.

Observe that

$$\varphi(cv_1 + c_2v_2) = c_1\varphi(v_1) + c_2\varphi(v_2) = c_1(\lambda v_1) + c_2(\lambda v_2) = \lambda(c_1v_1 + c_2v_2)$$

Therefore the space is closed under scalar multiplication and vector addition as desired. Note that $0$ is in the space by construction. \qed

Exercise 9. Let $V$ be a vector space over $F$ and let $\varphi$ be a linear transformation of the vector space $V$ to itself. Suppose for $i = 1, 2, \ldots, k$ that $v_i \in V$ is an eigenvector of $\varphi$ with eigenvalue $\lambda_i \in F$ and that all the eigenvalues $\lambda_i$ are distinct. Prove that $v_1, v_2, \ldots, v_k$ are linearly independent. Conclude that any linear transformation on an $n$-dimensional vector space has at most $n$ eigenvalues.

Proof. We proceed by induction over $k$. For $k = 1$, $v_1$ is trivially a linearly independent set since, by construction, $v_1 \neq 0$.

Assume that $\{v_1, v_2, \ldots, v_{k}\}$ as described in the problem is linearly independent and consider the set $\{v_1, v_2, \ldots, v_{k}\} \cup v_{k+1}$ as prescribed in the problem. Now, consider the linear combination $$c_1v_1 + c_2v_2 + \cdots + c_kv_k + c_{k+1}v_{k+1} = 0.$$ We will show that the only solution is $c_1 = c_2 = \cdots = c_k = c_{k+1} = 0$ by inspecting the following system

$$\begin{cases}
\lambda_1(c_1v_1 + c_2v_2 + \cdots + c_kv_k + c_{k+1}v_{k+1}) = \lambda_1(0) \\
\varphi(c_1v_1 + c_2v_2 + \cdots + c_kv_k + c_{k+1}v_{k+1}) = \varphi(0)
\end{cases}$$

equivalent to

$$\begin{cases}
c_1\lambda_1v_1 + c_2\lambda_1v_2 + \cdots + c_k\lambda_1v_k + c_{k+1}\lambda_1v_{k+1} = 0 \\
c_1\lambda_1v_1 + c_2\lambda_2v_2 + \cdots + c_k\lambda_kv_k + c_{k+1}\lambda_kv_{k+1} = 0
\end{cases}$$

If we subtract the second equation form the first one we have

$$(\lambda_1 - \lambda_2)c_2v_2 + (\lambda_1 - \lambda_3)c_3v_3 + \cdots + (\lambda_1 - \lambda_k)c_kv_k + (\lambda_1 - \lambda_{k+1})c_{k+1}v_{k+1} = 0.$$ Observe that the factors $(\lambda_1 - \lambda_i)$ are non zero since the eigenvalues are distinct. Furthermore, by the induction hypothesis, the only solution to the previous linear combination is the trivial solution. It is to say that $c_2 = c_3 = \cdots = c_k = c_{k+1} = 0$. Then, it must be the case that $c_1v_1 = 0$. But that implies that $c_1 = 0$ by construction. The result then follows.

For the second part, we know that any subset from a vector space of dimension $n$ has at most $n$ linearly dependent vectors. Since $\{v_1, v_2, \ldots, v_k, v_{k+1}\} \subseteq V$ the result follows. \qed

Section 11.2

Exercise 9. If $W$ is a subspace of the vector space $V$ stable under the linear transformation $\varphi$ (i.e., $\varphi(W) \subseteq W$), show that $\varphi$ induces linear transformations $\varphi|_W$ on $W$ and $\bar{\varphi}$ on the quotient
vector space $V/W$. If $\varphi \mid_W$ and $\tilde{\varphi}$ are non singular prove $\varphi$ is non singular. Prove the converse holds if $V$ has finite dimension and give a counterexample with $V$ infinite dimensional.

**Exercise 11.** Let $\varphi$ be a linear transformation form the finite dimensional vector space $V$ to itself such that $\varphi^2 = \varphi$.

a) Prove that $\text{Im}(\varphi) \cap \ker(\varphi) = 0$.

*Proof.* Assume there exist a non zero element $v$ in $\text{Im} \varphi \cap \ker \varphi$. Then,

i) for some $w$ in $V$, $\varphi(w) = v$, and

ii) $\varphi(v) = 0$.

We apply the transformation $\varphi$ to $\varphi(w) = v$ and see that

$$
\varphi(\varphi(w)) = \varphi(v) \\
\varphi^2(w) = 0 \\
\varphi(w) = 0 \\
v = 0
$$

a contradiction. \hfill \square

b) Prove that $V = \text{Im}(\varphi) \oplus \ker(\varphi)$.

*Proof.* Let $v = \varphi(v) + (v - \varphi(v))$. Observe that

$$
\varphi(v - \varphi(v)) = \varphi(v) - \varphi^2(v) = \varphi(v) - \varphi(v) = 0.
$$

Therefore $(v - \varphi(v)) \in \ker \varphi$ and, trivially, $\varphi(v) \in \text{Im} \varphi$. The result then follows. \hfill \square

c) Prove that there is a basis of $V$ such that the matrix of $\varphi$ with respect to this basis is a diagonal matrix whose entries are all 0 or 1.

*Proof.* We chose basis $B_1$ for $\text{Im} \varphi$ and $B_2$ for $\ker \varphi$ separately. Note that by part b) of the exercise $B_1 \cup B_2$ is a basis for $V$. Furthermore if $v \in B_1$, we know there exist $w$ such that $\varphi(w) = v$ for which

$$
\varphi(v) = \varphi^2(w) = \varphi(w) = v.
$$

This shows that the matrix representation of $\varphi$ restricted to the ordered basis $B_1$ is the identity matrix. Trivially, the matrix of $\varphi$ restricted to the kernel is the zero matrix. We conclude that the matrix representation of $\varphi$ under the ordered basis $B_1 \cup B_2$ is equal to

$$
\begin{pmatrix}
I & 0 \\
0 & 0
\end{pmatrix}
$$

as desired. \hfill \square

**Exercise 12.** Let $V = \mathbb{R}^2$, $v_1 = (1, 0)$, $v_2 = (0, 1)$, so that $v_1, v_2$ are a basis for $V$. Let $\varphi$ be the linear transformation of $V$ to itself whose matrix with respect to this basis is

$$
\begin{pmatrix}
2 & 1 \\
0 & 2
\end{pmatrix}.
$$

Prove that if $W$ is the subspace generated by $v_1$ then $W$ is stable under the action of $\varphi$. Prove that there is no subspace $W'$ invariant under $\varphi$ so that $V = W \oplus W'$.
Proof. Note that \( W = \text{span}(v_1) \). For arbitrary \( v \in \text{span}(v_1) \) we have that \( v = cv_1 \) (for \( c \) in the field of the vector space) then,

\[
\varphi(v) = \varphi(cv_1) = c\varphi(v_1) = 2cv_1 \in \text{span}(v_1).
\]

Hence, \( W \) is stable under \( \varphi \).

For the second part, assume that such \( W' \) exists. Then, it implies that the matrix of \( \varphi \) is diagonalizable. Observe that the characteristic polynomial of \( \varphi \) is \((2 - \lambda)^2\) and hence, \( \varphi \) has a unique eigenvalue. It can easily be show that the corresponding eigenspace of has dimension 1 and is spanned by \( v_1 \). Therefore the matrix is not diagonazible and we conclude that \( W' \) cannot exist. \( \square \)

**Exercise 36.** Let \( V \) be the 6-dimensional vector space over \( \mathbb{Q} \) consisting of polynomials in the variable \( x \) of degree at most 5. Let \( \varphi \) be the map of \( V \) to itself defined by \( \varphi(f) = x^2 f'' - 6xf' + 12f \).

a) Prove that \( \varphi \) is a linear transformation of \( V \) to itself.

**Proof.** Take \( p_1 \) and \( p_2 \) to be polynomials in the space and let \( \alpha \) and \( \beta \) be rational numbers. Then

\[
\varphi(\alpha p_1 + \beta p_2) = x^2 (\alpha p_1 + \beta p_2)'' - 6x (\alpha p_1 + \beta p_2)' + 12 (\alpha p_1 + \beta p_2)
= \alpha x^2 p_1'' + \beta x^2 p_2'' - \alpha 6xp_1' - \beta 6xp_2' + \alpha 12p_1 + \beta 12p_2
= \alpha (x^2 p_1'' - 6xp_1' + 12p_1) + \beta (x^2 p_2'' - 6xp_2' + 12p_2)
= \alpha \varphi(p_1) + \beta \varphi(p_2).
\]

It follows that \( \varphi \) is a linear transformation. Note that the space is preserved as all of the terms in the map are polynomials of degree at most 5. \( \square \)

b) Determine a basis for the image and for the kernel of \( \varphi \).

**Proof.** Let \( \{1, x, x^2, x^3, x^4, x^5\} \) of \( \mathbb{Q} \) be the basis for the space. Then the matrix representation of \( \varphi \) is

\[
\begin{pmatrix}
12 & 0 & 0 & 0 & 0 & 0 \\
0 & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}
\]

Clearly, the first three columns together with the last one are linearly independent. Therefore the image of the transformation has basis \( \{1, x, x^2, x^5\} \) (note that \( \dim \text{Im} \varphi = 3 \)). For the basis of the kernel it suffices to find two linearly independent vectors in the kernel (by the nullity-rank theorem). Observe that \( \varphi(x^4) = \varphi(x^3) = 0 \). Therefore, \( \{x^3, x^4\} \) is a basis for the kernel. \( \square \)

**Exercise 37.** Let \( V \) be the 7-dimensional vector space over the field \( F \) consisting of the polynomials in the variable \( x \) of degree at most 6. Let \( \varphi \) be the linear transformation of \( V \) to itself defined by \( \varphi(f) = f' \). For each of the fields below, determine a basis for the image and for the
Note. For all the following cases I proceed as in the previous exercise. I use the standard basis \{1, x, \ldots, x^7\} to construct a matrix for the transformation over the field, then I use the column space to see the basis of the image (denoted as \(I\)) and the basis of the kernel (denoted as \(K\)).

a) \(F = \mathbb{R}\)

Proof. By baby calculus, any polynomial of degree 6 has an antiderivative of degree 7. Hence \(I = \{1, x, x^2, x^3, x^4, x^5, x^6\}\). Trivially \(K = 1\) as any constant has derivative equal to 0.

b) \(F = \mathbb{F}_2\)

Proof. The corresponding matrix representation is

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

The linearly independent columns give \(I = \{1, x^2, x^4\}\) and the zero columns give \(K = \{1, x^3, x^6\}\).

c) \(F = \mathbb{F}_3\)

Proof. The corresponding matrix representation is

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

The linearly independent columns give \(I = \{1, x, x^3, x^4\}\) and the zero columns give \(K = \{1, x^3, x^6\}\).

d) \(F = \mathbb{F}_5\)

Proof. The corresponding matrix representation is

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

The linearly independent columns give \(I = \{1, x, x^2, x^3, x^5\}\) and the zero columns give \(K = \{1, x^5\}\).
Section 11.3

Exercise 2. Let \( V \) be the collection of polynomials with coefficients in \( \mathbb{Q} \) in the variable \( x \) of degree at most 5 with \( 1, x, x^2, \ldots, x^5 \) as basis. Prove that the following are elements of the dual space of \( V \) and express them as linear combinations of the dual basis.

**Note.** I will use the natural basis for the dual space; i.e.

\[
    v_i^*(v_j) = \begin{cases} 
    1 & i = j \\
    0 & i \neq j 
\end{cases}
\]

where \( v_i \) is the ordered element of the basis for \( V \).

a) \( E : V \to \mathbb{Q} \) defined by \( E(p(x)) = p(3) \).

**Proof.** Let \( \alpha p_1 + \beta p_2 \) be an element of \( V \). Note that

\[
    E(\alpha p_1 + \beta p_2) = (\alpha p_1 + \beta p_2)(3) = \alpha p_1(3) + \beta p_2(3) = \alpha E(p_1) + \beta E(p_2).
\]

Since \( E \) is linear, \( E \) is an element of the dual space of \( V \).

Note that \( E(x^i) = 3^i \). Then, under the standard dual basis

\[
    E = \sum_{i=0}^{5} 3^i v_i^*.
\]

\[ \square \]

b) \( \varphi : V \to \mathbb{Q} \) defined by \( \varphi(p(x)) = \int_0^1 p(t) \, dt \).

**Proof.** Let \( \alpha p_1 + \beta p_2 \) be an element of \( V \). Note that

\[
    \varphi(\alpha p_1 + \beta p_2) = \int_0^1 (\alpha p_1 + \beta p_2) \, dt = \alpha \int_0^1 p_1 \, dt + \beta \int_0^1 p_2 \, dt = \alpha \varphi(p_1) + \beta \varphi(p_2).
\]

Since \( \varphi \) is linear, \( \varphi \) is an element of the dual space of \( V \).

Note that \( \varphi(x^i) = \int_0^1 x^i \, dx = \frac{1}{i+1} \). Then, under the standard dual basis

\[
    \varphi = \sum_{i=0}^{5} \frac{1}{i+1} v_i^*.
\]

\[ \square \]

c) \( \varphi : V \to \mathbb{Q} \) defined by \( \varphi(p(x)) = \int_0^1 t^2 p(t) \, dt \).

**Proof.** Let \( \alpha p_1 + \beta p_2 \) be an element of \( V \). Note that

\[
    \varphi(\alpha p_1 + \beta p_2) = \int_0^1 t^2 (\alpha p_1 + \beta p_2) \, dt = \alpha \int_0^1 t^2 p_1 \, dt + \beta \int_0^1 t^2 p_2 \, dt = \alpha \varphi(p_1) + \beta \varphi(p_2).
\]

Since \( \varphi \) is linear, \( \varphi \) is an element of the dual space of \( V \).

Note that \( \varphi(x^i) = \int_0^1 x^2(x^i) \, dx = \frac{1}{i+3} \). Then, under the standard dual basis

\[
    \varphi = \sum_{i=0}^{5} \frac{1}{i+3} v_i^*.
\]

\[ \square \]
d) \( \varphi : V \to \mathbb{Q} \) defined by \( \varphi(p(x)) = p'(5) \).

Proof. Let \( \alpha p_1 + \beta p_2 \) be an element of \( V \). Note that

\[
\varphi(\alpha p_1 + \beta p_2) = (\alpha p_1 + \beta p_2)'(5) = \alpha p_1'(5) + \beta p_2'(5) = \alpha \varphi(p_1) + \beta \varphi(p_2).
\]

Since \( \varphi \) is linear, \( \varphi \) is an element of the dual space of \( V \).

Note that \( \varphi(x^i) = (x^i)'(5) = i \cdot 5^{i-1} \). Then, under the standard dual basis

\[
\varphi = \sum_{i=0}^{5} i \cdot 5^{i-1} v_i^*.
\]

Exercise 4. If \( V \) is infinite dimensional with basis \( A \), prove that \( A^* = \{ v^* | v \in A \} \) does not span \( V^* \).

Proof. Let \( \varphi : V \to F \) be defined as \( \varphi(v_i) = 1 \) for all \( v_i \in A \). Then,

\[
\varphi = \sum_{i \in A} v_i^*
\]

can only have finitely many non zero terms. In other words

\[
\varphi = \sum_{i \in \mathcal{B}} v_i^*
\]

for some finite set \( \mathcal{B} \). Note that this implies that \( \varphi(w) = 0 \) for \( w \notin A \), therefore \( \varphi \) is not in the span of \( A^* \).

Section 11.4

Exercise 6. (Minkowski’s Criterion) Suppose \( A \) is an \( n \times n \) matrix with real entries such that the diagonal elements are all positive, the off-diagonal elements are all negative and the row sums are all positive. Prove that \( \det A \neq 0 \).

Proof. Assume \( \det(A) = 0 \). Then, there exists a non zero solution for the system \( AX = 0 \).

Let \( x_m \) be the maximum coordinate element of the vector \( X \) in absolute value. Then, we have the following.

\[
\left| \sum_{j=1}^{n} a_{i,j} \cdot x_j \right| \leq \sum_{j=1}^{n} |a_{i,j} \cdot x_j| \leq \sum_{j=1}^{n} |a_{i,j}| \cdot |x_j| \leq \sum_{j=1}^{n} |a_{i,j}| \cdot |x_m|
\]

Taking the last inequality, we fix \( i \) to be equal to \( m \) and we multiply by \(-1\) to get

\[
-\sum_{j=1}^{n} |a_{m,j}| \cdot |x_j| \geq -\sum_{j=1}^{n} |a_{m,j}| \cdot |x_m|
\]

Now we add \( 2|a_{m,m}| \cdot |x_m| \) and simplify to

\[
|a_{m,m}| \cdot |x_m| - \sum_{j=1}^{n} j \neq m |a_{m,j}| \cdot |x_j| \geq |a_{m,m}| \cdot |x_m| - \sum_{j=1}^{n} j \neq m |a_{m,j}| \cdot |x_m|
\]
Note that the right hand side of the inequality is equal to $|x_j| \sum_{j=1}^n a_{m,j}$ which on its own is greater than 0 by construction. It is to say $|a_{m,m}| \cdot |x_m| - \sum_{j=1, j \neq m}^n |a_{m,j}| \cdot |x_j| \geq |x_j| \sum_{j=1}^n a_{m,j} > 0$

or simply

$|a_{m,m}| \cdot |x_m| - \sum_{j=1, j \neq m}^n |a_{m,j}| \cdot |x_j| > 0.$

The left hand side can be compacted by means of the reverse triangle inequality, thus

$\left| a_{m,m} \cdot x_m - \sum_{j=1, j \neq m}^n |a_{m,j}| \cdot |x_j| \right| \geq |a_{m,m}| \cdot |x_m| - \sum_{j=1, j \neq m}^n |a_{m,j}| \cdot |x_j| > 0.$

Lastly, the left side of the inequality reduces to $\sum_{j=1}^n a_{m,j} \cdot x_j$ and we have

$\sum_{j=1}^n a_{m,j} \cdot x_j > 0.$

Which is a contradiction to $AX = 0$. □

Section 11.5

Exercise 13. Let $F$ be any field in which $-1 \neq 1$ and let $V$ be a vector space over $F$. Prove that $V \otimes_F V = S^2(V) \oplus \Lambda^2(V)$ i.e., that every 2-tensor may be written uniquely as a sum of a symmetric and an alternating tensor.

Proof. First, observe that $\dim S^2(V) = \frac{n(n+1)}{2}$ and that $\dim \Lambda^2(V) = \frac{n(n-1)}{2}$. Hence, $\dim S^2(V) \oplus \Lambda^2(V) = n^2 = \dim V \otimes_F V$. The result follows if we prove that the spaces $S^2(V)$ and $\Lambda^2(V)$ intersect trivially. To that end, assume $v \in S^2(V) \cap \Lambda^2(V)$. Then, by definition

$$\begin{cases} 
\sigma v = v \\
\sigma v = \text{sign}(\sigma)v
\end{cases} \iff v = \text{sign}(\sigma)v \iff v(1 - \text{sign}(\sigma)) = 0$$

Note that $\text{sign}(\sigma)$ is necessarily equal to 1 since we are working with the symmetric group $S_2$. The above equation implies that $v(1-1) = 0$. By construction $1-1 \neq 0$, and thus $v$ is forced to be equal to 0, as desired. □