Selected exercises from *Abstract Algebra* by *Dummit and Foote* (3rd edition).

Bryan Félix

Abril 12, 2017

Section 4.1

**Exercise 1.** Let $G$ act on the set $A$. Prove that if $a, b \in A$ and $b = ga$ for some $g \in G$, then $G_b = gG_ag^{-1}$. Deduce that if $G$ acts transitively on $A$ then the kernel of the action is $\bigcap_{g \in G} gG_ag^{-1}$.

**Proof.** For the first part we use the usual containment criterion.

i) We show that $g^{-1}G_bg \subseteq G_a$ (equivalently $G_b \subseteq gG_ag^{-1}$).

Let $x \in g^{-1}G_b$, then

\[
x = g^{-1}\tilde{b}g \quad \text{for some } \tilde{b} \in G_b
\]

\[
x \cdot a = g^{-1}\tilde{b}g \cdot a
\]

\[
= g^{-1}\tilde{b}(g \cdot a)
\]

\[
= g^{-1}\tilde{b} \cdot (b)
\]

\[
= g^{-1}(\tilde{b} \cdot b)
\]

\[
= g^{-1} \cdot b
\]

\[
= a
\]

Therefore $G_b \subseteq gG_ag^{-1}$.

ii) Now we show $gG_ag^{-1} \subseteq G_b$.

Let $x \in gG_ag^{-1}$, then

\[
x = g\bar{a}g^{-1} \quad \text{for some } \bar{a} \in G_a
\]

\[
x \cdot b = g\bar{a}g^{-1} \cdot b
\]

\[
= g\bar{a}(g^{-1} \cdot b)
\]

\[
= g\bar{a} \cdot (a)
\]

\[
= g(\bar{a} \cdot a)
\]

\[
= g \cdot a
\]

\[
= b
\]

Hence, $gG_ag^{-1} \subseteq G_b$ as desired.
For the second part, recall that the kernel of the action is given by the intersection of the stabilizers of the elements in $A$. Since the action is transitive, we have that the orbit of $a$ is equal to the entire set $A$. Then, by means of the previous part we have

$$\bigcap_{g \in G} gG_ag^{-1} = \bigcap_{g \in G} G_{ga} = \bigcap_{a \in A} G_a$$

as desired.

**Exercise 2.** Let $G$ be a permutation group on the set $A$ (i.e., $G \leq S_A$), let $\sigma \in G$ and let $a \in A$. Prove that $\sigma G_a \sigma^{-1} = G_{\sigma(a)}$. Deduce that if $G$ acts transitively on $A$ then

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = 1$$

**Proof.** The first part is trivial using exercise 1. Observe that $a$ and $\sigma(a)$ are elements of $A$, with the identity

$$\sigma(a) = \sigma \cdot a.$$  

For the second part, observe that if the action is transitive, then there is only one orbit. Using Lagrange’s we see that

$$\frac{|G|}{|G_a|} = |O_a| = |G|$$

therefore $|G_a| = 1$ for any $a \in G$. Hence $G_a = 1$ for any $a \in G$. It follows that

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = \bigcap_{\sigma \in G} G_{\sigma(a)} = \bigcap_{\sigma \in G} 1 = 1$$

as desired. 

**Exercise 3.** Assume that $G$ is abelian, transitive subgroup of $S_A$. Show that $\sigma(a) \neq a$ for all $\sigma \in G - 1$ and all $a \in A$. Deduce that $|G| = |A|$.

**Proof.** If $G$ is abelian we have

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = \bigcap_{\sigma \in G} G_a = G_a = 1$$

as desired. For the second part, we use Lagrange’s and we have that

$$[G : G_a] = |O_a| = |G|$$

as desired.

**Section 4.2**

**Exercise 7.** Let $Q_8$ be the quaternion group of order 8

a) Prove that $Q_8$ is isomorphic to a subgroup of $S_8$

**Proof.** This is trivial using Cayley’s Theorem (Corollary 4).
b) Prove that $Q_8$ is not isomorphic to a subgroup of $S_n$ for any $n \leq 7$.

Proof. It suffices to show that $Q_8$ is not isomorphic to any subgroup of $S_7$ (since $S_1 < S_2 < \cdots < S_6 < S_7$).

Let $Q_8$ act on a set $A$ of order 7. Then we inspect the order of the orbit and stabilizer of an arbitrary element $a \in A$. From Lagrange’s, the order of the stabilizer divides the order of the group $Q_8$. Therefore the order is either 1, 2, 4 or 8. If the order of the stabilizer equals 1 then, from

$$[G : G_a] = |O_a|$$

we have that the orbit has order 8. This is a clear contradiction since the set $A$ has seven elements. Therefore the order of the stabilizer is either 2, 4 or 8. In any case we can see form the lattice of $Q_8$ that any such subgroup contains the group $\langle -1, 1 \rangle$. It follows that the kernel of the action

$$\ker(\text{action}) = \bigcap_{a \in A} G_a$$

contains the subgroup $\langle -1, 1 \rangle$. Therefore the action is not faithful, then not injective, and hence, no isomorphism exists.

Exercise 9. Prove that if $p$ is a prime and $G$ is a group of order $p^\alpha$ for some $\alpha \in \mathbb{Z}^+$, then every subgroup of index $p$ is normal in $G$. Deduce that every group of order $p^2$ has a normal subgroup of order $p$.

Proof. We assume that $\alpha \geq 2$ otherwise the group is cyclic and the only subgroup of index $p$ is the identity (which is normal).

We know that groups of order $p^\alpha$ are nilpotent. The following lemma is taken from Isaacs Algebra (2008):

Lemma. The following are equivalent:

i) $G$ is a nilpotent group.

ii) Every maximal proper subgroup of $G$ is normal.

The result then follows for all $\alpha \geq 2$. For groups of order $p^2$ we have that $G$ is cyclic, then it suffices to show that a subgroup of order $p$ exists. Let $x$ be the generator of $G$, then $\langle x^2 \rangle$ is a subgroup of $G$ of order $p$.

Section 4.3

Exercise 6. Assume $G$ is a non-abelian group of order 15. Prove that $Z(G)=1$.

Proof. We inspect the order of $Z(G)$. Since $Z(G)$ is a subgroup of $G$, it’s order divides 15. Then we have four cases:

i) The order of $Z(G)$ is equal to 1. Then the conclusion is trivial.

ii) The order of $Z(G)$ is equal to 3. Then $[G : Z(G)] = 5$ and therefore $G/Z(G)$ is cyclic. By exercise 36 on section 3.1 we have that $G$ is abelian arriving at a contradiction.
iii) The order of $Z(G)$ is equal to 5. 
Like the previous case $G/Z(G)$ is cyclic and then $G$ is abelian.

iv) The order of $Z(G)$ is equal to 15. 
Then $G$ is abelian by definition. A clear contradiction.

Exercise 30. If $G$ is a group of odd order, prove for any non identity element $x \in G$ that $x$ and $x^{-1}$ are not conjugate in $G$.

Proof. We proceed by contradiction and we assume that $x^{-1}$ is a conjugate of $x$. Then, we look at the action of $G$ with itself by conjugation and inspect the orbit $O_x$ of $x$.
If the only elements of the orbit are $x$ and $x^{-1}$ then the order of $O_x$ equals two and we have a contradiction (by Lagrange’s theorem) since 2 does not divide $|G|$. Then, there must be $y \in O_x$ such that $y \neq x$.
We make a remark and note that $y \neq 1$ either. Otherwise (by the definition of conjugate elements)
$$x = g1g^{-1}$$
$$x = 1.$$ is a contradiction.
Now take $y$ and observe that
$$x = gyg^{-1}$$
$$x^{-1} = gy^{-1}g^{-1}$$
Therefore $y^{-1}$ is in the same orbit as $x^{-1}$ and hence both $y$ and $y^{-1}$ are in $O_x$. Again, we observe that the order of $O_x$ is even, and we arrive at a contradiction again.
By the principle of indefinite exhaustion the existence of $O_x$ contradicts the assumption as desired.

Section 4.4

Exercise 7. If $H$ is the unique subgroup of a given order in a group $G$ prove $H$ is characteristic on $G$.

Proof. Let $\sigma$ be an element of Aut($G$). Recall that $\sigma$ is an isomorphism $\sigma : G \rightarrow G$. By properties of isomorphisms, group orders are preserved; i.e.
$$|H| = |\sigma(H)|$$
Since $H$ is the unique subgroup of $G$ of order $|H|$, this forces $\sigma$ to map $H$ to itself.
Remark. It is not necessary that $\sigma(h) = h$ for $h \in H$.

Exercise 8. Let $G$ be a group with subgroups $H$ and $K$ with $H \leq K$.

a) Prove that if $H$ is characteristic in $K$ and $K$ is normal in $G$ then $H$ is normal in $G$. 

Proof. Let $G$ act on $K$ by conjugation and let $\sigma_g$ be the associated permutation of a fixed element in $G$ acting on $K$. Observe that since $K$ is normal $\sigma_g(K) = K$ for all $g \in G$. Furthermore $\sigma_g$ is an isomorphism of $K$, therefore $\sigma_g \in \text{Aut}(K)$. Then, since $H$ is characteristic in $K$ we have that $\sigma(H) = H$ for all $g \in G$. Equivalently $gHg^{-1} = H$ as desired. 

b) Prove that if $H$ is characteristic in $K$ and $K$ is characteristic in $G$ then $H$ is characteristic in $G$.

Proof. Let $\sigma$ be an element in $\text{Aut}(G)$. Since $K$ is characteristic in $G$, $\sigma(K) = K$. Furthermore $\sigma$ restricted to $K$ is an isomorphism of $K$, therefore $\sigma|_K \in \text{Aut}(K)$. It is easy to see that

$$\sigma|_K(H) = H$$

and therefore $\sigma(H) = H$ as desired.

Exercise 9. If $r, s$ are the usual generators for the dihedral group $D_{2n}$, use the preceding two exercises to deduce that every subgroup of $\langle r \rangle$ is normal in $D_{2n}$.

Proof. We will show that $\langle r \rangle$ is normal in $D_{2n}$ and then show that any subgroup of $\langle r \rangle$ is characteristic.

For the first part note that the only generator outside $\langle r \rangle$ is $s$. Then it suffices to show that $\langle r \rangle = \langle srs^{-1} \rangle$. Observe that

$$\langle srs^{-1} \rangle = \langle r^{-1} \rangle$$

$$= \langle r \rangle$$

as desired. Therefore $\langle r \rangle$ is normal in $D_{2n}$.

For the second part, recall that a group is cyclic if and only if no two (distinct) subgroups have the same order. Therefore, by problem 7, the subgroups of $\langle r \rangle$ are characteristic. The result then follows.

Exercise 16. Prove that $(\mathbb{Z}/24\mathbb{Z})^\times$ is an elementary abelian group of order 8.

Proof. It suffices to show that for every number $n < 24$ relative prime to 24, $n^2 \equiv 1 \pmod{24}$. Observe that

$$1^2 = 1 \equiv 1 \pmod{24}$$
$$5^2 = 25 \equiv 1 \pmod{24}$$
$$7^2 = 49 \equiv 1 \pmod{24}$$
$$11^2 = 121 \equiv 1 \pmod{24}$$
$$13^2 = 169 \equiv 1 \pmod{24}$$
$$17^2 = 289 \equiv 1 \pmod{24}$$
$$19^2 = 361 \equiv 1 \pmod{24}$$
$$23^2 = 529 \equiv 1 \pmod{24}$$

Hence, $(\mathbb{Z}/24\mathbb{Z})^\times$ is an elementary abelian group of order 8.
Section 4.5

Exercise 13. Prove that a group of order 56 has a normal Sylow p-subgroup for some prime p dividing its order.

Proof. Using Sylow’s theorems we see that

\[ n_2 = 1 \text{ or } 7 \]

and

\[ n_7 = 1 \text{ or } 8. \]

We proceed by contradiction and we assume that neither of the Sylow subgroups is normal. Therefore we necessarily have \( n_2 = 7 \) and \( n_7 = 8 \). Since the Sylow 7-subgroups only intersect at the identity we have \( 8(7−1) = 48 \) non-identity elements of order 7 in \( G \). Observe that none of these elements can belong to the Sylow 2-subgroups by Lagrange’s. Then we are left with 8 elements belonging to the 8 distinct Sylow 2-subgroups. This is impossible and we have our contradiction.

Exercise 16. Let \(|G| = pqr\), where \(p, q,\) and \(r\) are primes with \(p < q < r\). Prove that \(G\) has a normal Sylow subgroup for either \(p, q,\) or \(r\).

Proof. We inspect the values of \(n_r\) and \(n_q\). By the Sylow theorems we must have that \(n_r\) satisfies:

\[ n_r \equiv 1 \mod r \]

and

\[ n_r \mid pq. \]

The latter restricts the options to \(n_r\) being either 1, \(p, q\) or \(pq\). Note that if \(n_r = 1\) then the Sylow subgroup is normal and we are done. Otherwise note that the assumption \(p > q > r\), forces \(n_r\) to be equal to \(pq\) (otherwise the congruence is not satisfied).

Likewise we inspect the possible values of \(n_q\) and conclude that \(n_q\) is either \(r\) or \(pr\).

Now, we make the standard count element. The \(pq\) Sylow r-subgroups contribute with \(pq(r−1)\) elements while the Sylow q-subgroups contribute with at least \(r(q−1)\) elements. In total we have

\[
pq(r−1) + r(q−1) = pq(r−1) + p(q−1) = pqr - pq + pq - p = pqr - p
\]

Recall that the order of \(G\) is \(pqr\). Therefore we only have \(p\) elements to distribute to the renaming Sylow subgroups. This forces the uniqueness of the Sylow p-subgroup, making is a normal subgroup of \(G\) as desired.

Exercise 30. How many elements of order 7 must there be in a simple group of order 168?

Solution. Note that 168 = 2³ · 3 · 7. If we inspect the number of Sylow 7-subgroups \(n_7\) we see that

\[ n_7 \equiv 1 \mod 7 \]

and

\[ n_7 \mid 24 \]
The latter restricts \( n_7 \) to either 1, 2, 3, 4, 6, 8, 12 or 24. Together with the congruence and the fact that the group is simple we have that \( n_7 = 8 \). Since these Slow 7-subgroups are of prime order, they all intersect only at the identity, therefore we have

\[
8(7 - 1) = 48
\]

elements of order 7 in the group. \( \square \)

**Exercise 33.** Let \( P \) be a normal Sylow \( p \)-subgroup of \( G \) and let \( H \) be any subgroup of \( G \). Prove that \( P \cap H \) is the unique Sylow \( p \)-subgroup of \( H \).

**Proof.** Since \( P \) is normal in \( G \), \( H \) is a subgroup of \( N(P) \) and we may use the second isomorphism theorem. Then, \( H \cap P \) is normal in \( H \) and by Corollary 20 \( H \cap P \) is the unique Sylow subgroup of \( H \). \( \square \)

**Exercise 34.** Let \( P \in \text{Syl}_p(G) \) and assume \( N \trianglelefteq G \). Use the conjugacy part of Sylow’s Theorem to prove that \( P \cap N \) is a Sylow \( p \)-subgroup of \( N \). Deduce that \( PN/N \) is a Sylow \( p \)-subgroup of \( G/N \).

**Proof.** Take any Sylow \( p \)-subgroup \( H \) of \( N \) and observe that \( H \) is a \( p \)-subgroup in \( G \), therefore there exist \( g \in G \) such that

\[
H \leq gPg^{-1}.
\]

Furthermore \( H \) is also a subgroup of \( gNg^{-1} \) (by the normality of \( N \)). Then

\[
H \leq (gPg^{-1}) \cap (gNg^{-1})
\]

and

\[
gHg^{-1} \leq P \cap N.
\]

Note that both \( gHg^{-1} \) and \( P \cap N \) are \( p \)-subgroups in \( G \). Since \( |gHg^{-1}| = |H| \) and \( H \) is a Sylow \( p \)-subgroup of \( N \) it follows that \( P \cap N \) has \( p \)-power order at least as large as \( H \). Therefore \( P \cap N \) is a Sylow \( p \)-subgroup of \( N \).

For the second part we use the second isomorphism theorem. Since \( N \) is normal \( P \leq N(N) \) and therefore \( PN \) is a subgroup of \( G \). We only need to show that \( p \) does not divide the order of the index

\[
[G/N : PN/N].
\]

By the second isomorphism theorem \( PN/N \cong P/P \cap N \), then

\[
[G/N : PN/N] = [G/N : P/P \cap N] = \frac{|G| |P \cap N|}{|P||N|}.
\]

By assumption \( (P \in \text{Syl}_p(G)) \) \( p \) does not divide \( \frac{|G|}{|P|} \) and by the first part of the problem \( (P \cap N \in \text{Syl}_p(N)) \) \( p \) does not divide \( \frac{P \cap N}{N} \). Thus, \( p \) does not divide \( [G/N : PN/N] \) as desired. \( \square \)