Study guide for the final exam

Math 2374, Fall 2003

1. Basic coordinate and vector geometry (chapter 1, section 2.1, section 3.1)
   (a) Key items: quadric surfaces, cylindrical and spherical coordinates, computing 2×2 and 3×3 determinants, understanding and computing dot products and cross products, equations and parametrization of lines and planes, derivatives of (vector-valued) functions of one variable, magnitudes and angles between vectors in $\mathbb{R}^n$, level curves, and plots of vector fields.
   (b) Relevance to calculus: this material underlies the work of the class. Mastery of these basics is needed to do the rest.
2. Linear algebra
   (a) Basic matrix properties and manipulations (section 2.2)
      - Key items: matrix-vector and matrix-matrix products, symmetric matrices, invertible matrices
   (b) Linear transformations (section 2.3)
      i. Key idea: The one-to-one correspondence between linear transformations (linear functions) and matrices.
      ii. Supporting concept: vectors as column matrices.
      iii. Important conclusion: linear functions have properties inherited from matrices.
      iv. Relevance to calculus: the Jacobian matrix and its associated linear function
3. Quadratic forms (section 2.5)
   (a) Sample HW problems: Section 2.5, #2, 8, 15
   (b) Key idea 1: The one-to-one correspondence between quadratic forms and symmetric matrices
   (c) Key idea 2: Categorizing definiteness of quadratic forms and symmetric matrices, i.e., positive definite, negative definite, indefinite
   (d) Methods: Determine definiteness by inspection and using Sylvester's theorem
   (e) Relevance to calculus: the Hessian matrix and Hessian form, finding local minima and maxima.
4. Derivatives
   (a) Partial derivatives (section 3.4)
      i. Key items: understand and compute partial derivatives
      ii. Methods: limit definition and one-variable calculus techniques
   (b) The total derivative (section 3.5)
      i. Key idea 1: the total derivative is represented by the Jacobian matrix and its associated linear function.
      ii. Key idea 2: use the total derivative to write a linear approximation (differential approximation) of a function $\mathbf{f}$ near a point $\mathbf{a}$.

   (a) The chain rule (section 3.6)
      i. Key idea: Gives the total derivative of a composition of functions.
      ii. Key formula: $J_{\mathbf{g} \circ \mathbf{f}}(\mathbf{a}) = J_{\mathbf{g}}(\mathbf{f}(\mathbf{a})) J_{\mathbf{f}}(\mathbf{a})$
      iii. Note: Formulas for partial derivatives can be derived from above formula, but be careful to evaluate partials of $\mathbf{g}$ at the point $\mathbf{f}(\mathbf{a})$.
5. Gradient, directional derivative, divergence, and curl (sections 4.1 and 4.2)
   (a) Gradient key ideas: applies to scalar-valued functions only, points in direction of greatest increase, is normal to level curves and level surfaces, denoted ∇f.

(b) Directional derivative key ideas: applies to scalar-valued functions only, is like a partial derivative taken in any direction, is a number representing the slope in that direction, $D_{\mathbf{u}}f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot u$ and $D_{\mathbf{u}}f(\mathbf{a}) = \|\nabla f(\mathbf{a})\| \cos\theta$.

(c) Divergence key ideas: applies to vector-valued functions only, measures outflow per unit volume of fluid flow, denoted $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F}$.

(d) Curl key ideas: applies to vector-valued functions only, measures rotation of fluid flow, denoted $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F}$

6. Taylor's theorem and local extrema (sections 4.3 and 4.4)

   (a) Sample HW problems: Section 4.3 #10, Section 4.4 #19, 21

   (b) Key construct 1: the Hessian matrix $H_f(\mathbf{a})$ and the Hessian form $h(\mathbf{x}) = \mathbf{x}^T H_f(\mathbf{a})\mathbf{x}$.

   (c) Key construct 2: 2^{nd}-degree Taylor polynomial $f(\mathbf{x}) \approx f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a}) + \frac{1}{2}(\mathbf{x}-\mathbf{a})^T H_f(\mathbf{a})(\mathbf{x}-\mathbf{a})$

   (d) Important application: critical points $\nabla f(\mathbf{a}) = \mathbf{0}$ can be classified as extrema based on definiteness of $H_f(\mathbf{a})$.

7. Paths (parameterized curves) (sections 5.1 and 5.2)

   (a) Given a simple curve, find a parametrization.

      i. For simple curves such as line segments and circle segments.

      ii. Can parameterize in two directions (orientations). (In parallel to surfaces, could think of unit tangent vector $\mathbf{T} = \mathbf{f}'(t)/\|\mathbf{f}'(t)\|$ as specifying direction.)

   (b) Find arclength of a parametrized curve

      i. Key idea: arclength element of $\mathbf{x} = \mathbf{f}(t)$ is $dL = \|\mathbf{f}'(t)\|dt$.

      ii. Formula: $L(C) = \int_a^b \|\mathbf{f}'(t)\|dt$.

   (c) Line integrals (path integrals)

      i. Line integrals of scalar-valued functions

         A. Key idea: Integrate scalar function $u(\mathbf{x})$ along curve (i.e., $u(\mathbf{f}(t))$) using above dL.

         B. Formula: $\int_C u\,dL = \int_a^b u(\mathbf{f}(t))\|\mathbf{f}'(t)\|dt$

      ii. Line integrals of vector-valued functions

         A. Key idea: Integrate tangent component of $\mathbf{F}(\mathbf{x})$ along curve (i.e. $\mathbf{F}(\mathbf{f}(t)) \cdot \mathbf{T}$) using above dL.

         B. Formula: $\int_C \mathbf{F} \cdot d\mathbf{x} = \int_C \mathbf{F} \cdot \mathbf{T}\,dL = \int_a^b \mathbf{F}(\mathbf{f}(t)) \cdot \mathbf{f}'(t)dt$.

8. Parameterized surfaces (section 5.5 and 5.6)

   (a) Given a surface, find a parameterization

      i. Key surfaces: spheres, cylinders, planes, any surface of form $z = h(x, y)$.

      ii. Unit normal $\mathbf{n} = \frac{\partial \mathbf{f}}{\partial s} \times \frac{\partial \mathbf{f}}{\partial t} \Big/ \left\|\frac{\partial \mathbf{f}}{\partial s} \times \frac{\partial \mathbf{f}}{\partial t}\right\|$ specifies orientation.

   (b) Find surface area of a parameterized surface

      i. Key idea: surface area element of $\mathbf{x} = \mathbf{f}(s,t)$ is $d\sigma = \left\|\frac{\partial \mathbf{f}}{\partial s} \times \frac{\partial \mathbf{f}}{\partial t}\right\| ds\,dt$

      ii. Formula: $\sigma(M) = \int\int_R \left\|\frac{\partial \mathbf{f}}{\partial s} \times \frac{\partial \mathbf{f}}{\partial t}\right\| ds\,dt$.

   (c) Surface integrals

      i. Surface integrals of scalar-valued functions

         A. Key idea: Integrate scalar function $g(\mathbf{x})$ across surface (i.e., $g(\mathbf{f}(s,t))$) using above $d\sigma$.

         B. Formula: $\int\int_M g\,d\sigma = \int\int_R g(\mathbf{f}(s,t)) \left\|\frac{\partial \mathbf{f}}{\partial s} \times \frac{\partial \mathbf{f}}{\partial t}\right\| ds\,dt$.

      ii. Surface integrals of vector-valued functions

         A. Key idea: Integrate normal component of $\mathbf{F}(\mathbf{x})$ across surface (i.e., $\mathbf{F}(\mathbf{f}(s,t)) \cdot \mathbf{n}$) using above $d\sigma$.

         B. Formula: $\int\int_M \mathbf{F} \cdot \mathbf{n}\,d\sigma = \int\int_R \mathbf{F}(\mathbf{f}(s,t)) \cdot \left(\frac{\partial \mathbf{f}}{\partial s} \times \frac{\partial \mathbf{f}}{\partial t}\right) ds\,dt$.

9. Double and triple integrals (sections 5.3 and 5.4)
   (a) Key idea: although defined by Riemann sums over rectangles (double integrals) or boxes (triple integrals), these integrals can be computed through iterated integrals.
   (b) One trick: computing bounds for iterated integrals, especially for the different orders of integration.
   (c) Remember: outer limits must be constant; inner limits can depend only on variables from the outside integral(s).
10. Change of variables (sections 5.7 and 5.8)
   (a) In double integrals
      i. Important special case: polar coordinates
      ii. Key idea: Evaluate integral in new region over new coordinates with new area measure $dA = \left|\frac{\partial f(s,t)}{\partial(s,t)}\right| ds\,dt$.
      iii. Formula: $\int\int_R g(x,y)dx\,dy = \int\int_{R^*} g(\mathbf{f}(s,t)) \left|\frac{\partial f(s,t)}{\partial(s,t)}\right| ds\,dt$.
   (b) In triple integrals
      i. Important special cases: cylindrical coordinates, spherical coordinates
      ii. Key idea: Evaluate integral in new region over new coordinates with new volume measure $dV = \left|\frac{\partial f(s,t,u)}{\partial(s,t,u)}\right| ds\,dt\,du$.
      iii. Formula: $\int\int_S g(x,y,z)dx\,dy\,dz = \int\int_{S^*} g(\mathbf{f}(s,t,u)) \left|\frac{\partial f(s,t,u)}{\partial(s,t,u)}\right| ds\,dt\,du$.
11. The fundamental theorem for path integrals (section 6.1)
   (a) Key idea: test if a vector field is path-independent (conservative). If it is, your life got a lot easier (that is, if you're trying to compute a line integral of the vector field).
   (b) Fact: if a vector field $\mathbf{F}$ is path-independent, then
      i. its line integral depends only on the endpoints (so is zero over closed curves)
      ii. $\mathbf{F} = \nabla f$
      iii. $\int_C \mathbf{F} \cdot d\mathbf{x} = f(\mathbf{b}) - f(\mathbf{a})$, where $\mathbf{a}$ and $\mathbf{b}$ are the endpoints of the path.
   (c) Test for path-independence: on a **simply connected** domain, $\mathbf{F}$ is path-independent if and only if its Jacobian matrix is symmetric.
      i. In 2D, the symmetric Jacobian condition is $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 0$.
      ii. In 3D, the symmetric Jacobian condition is $\operatorname{curl}\mathbf{F} = \mathbf{0}$.
   (d) Don't forget the consequence of having a hole through the domain.
12. Green's Theorem (section 6.2)
   (a) Key idea: If computing a line integral of a vector field $\mathbf{F}$ over a closed curve in 2D, you can convert it to a double integral (if $\mathbf{F}$ is defined in the whole interior of the curve).
   (b) Formula: $\displaystyle\int_{\partial R} \mathbf{F} \cdot d\mathbf{x} = \int\int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$.
13. Stokes' Theorem (section 6.3)
   (a) Key idea: to calculate circulation of $\mathbf{F}$ around closed curve C, you can choose any surface with boundary C and calculate flux integral of $\operatorname{curl}\mathbf{F}$ over surface.
   (b) Need positively oriented boundary: walk on positive side of surface near boundary and surface is on left (CCW boundary viewed from positive side). Positive side is side with normal.
   (c) Formula: $\int_{\partial M} \mathbf{F} \cdot d\mathbf{x} = \int\int_M \operatorname{curl}\mathbf{F} \cdot \mathbf{n}\, d\sigma$.
14. Divergence Theorem (section 6.4)
   (a) Key idea: to calculate flux of $\mathbf{F}$ across closed surface M from inside to outside, instead calculate the triple integral of $\operatorname{div}\mathbf{F}$ over solid enclosed by M.
   (b) Formula: $\int\int_{\partial S} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int\int\int_S \operatorname{div}\mathbf{F}\, dV$.