Sets and Functions

Recall that a **set** is an *unordered* collection of things.

A set can be described as a comma-separated **list** enclosed by *braces* (though the apparent ordering is *not intrinsic*), like

$$\{1, 2, 3\}$$

Since order does not matter,

$$\{1, 2, 3\} = \{3, 1, 2\} = \{2, 1, 3\} = \text{ etc.}$$

Also, repeating an element does not do anything:

$$\{1, 2, 3\} = \{1, 1, 1, 2, 2, 3, 2\}$$

The things x in a set S are the **elements** of the set. Notation is $x \in S$ or $S \ni x$. Examples:

$$1 \in \{1, 2, 3\}$$

$$\{1, 2, 3\} \ni 3$$

$$4 \notin \{1, 2, 3\}$$

The **union** $A \cup B$ of two sets consists of the elements lying in either set. Example:

$$\{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$$

The **intersection** $A \cap B$ of two sets consists of the elements lying in both. Example:

$$\{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$$

Two sets A, B are **disjoint** if they have no elements in common, that is, if their intersection is the empty set

$$\phi = \{\}$$

Sets can have elements which are themselves sets. Example:

$$\{1, 2, 3, \{1, 2\}, \{\{1\}\}\}$$

has elements

$$1, 2, 3, \{1, 2\}, \{\{1\}\}$$

The **(cartesian) product** $A \times B$ of two sets is the set of **ordered pairs** (a, b) with $a \in A$ and $b \in B$.

Intuitively, a **function** f from a set A to a set B is a thing which accepts inputs from A and produces outputs in B. The notation

$$f : A \to B$$

means f is a function from a set A to a set B.

(This usage is standard, so other uses of arrows are unwise.)

A function $f : A \to B$ **must**: accept as input every element of the set A, produce the same output for the same input, produce outputs in the set B, and not fail to produce an output.

The formal definition of a function $f : A \to B$ is by specifying it by its **graph**: it is a subset of the cartesian product $A \times B$ such that for every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$.

In practice we *describe* a function by either

- Listing inputs and corresponding outputs
- Giving a formula or procedure to produce output for a given input

Other terminology: a **look-up table** for a function $f : A \to B$ is a **list** of outputs corresponding to all possible legal inputs of f.

Example: to describe a function

$$f : \{1, 2, 3\} \to \{7, 8\}$$

we must tell exactly 3 things, namely $f(1)$, $f(2)$, and $f(3)$. We do *not* have to give a formula. For example,

$$f(1) = 7 \quad f(2) = 7 \quad f(3) = 8$$

is a legitimate description of *one* particular function f.

As a set of ordered pairs, this function (its graph) is

$$f = \{(1, 7), (2, 7), (3, 8)\}$$

Functions and formulas are different things: a function *may* be given by a formula, but not necessarily.

We can list *all* functions

$$f : \{1, 2, 3\} \to \{7, 8\}$$

by telling (for each) the output for each input:

case 1:	$f(1)=7$	$f(2)=7$	$f(3)=7$
case 2:	$f(1)=7$	$f(2)=7$	$f(3)=8$
case 3:	$f(1)=7$	$f(2)=8$	$f(3)=7$
case 4:	$f(1)=7$	$f(2)=8$	$f(3)=8$
case 5:	$f(1)=8$	$f(2)=7$	$f(3)=7$
case 6:	$f(1)=8$	$f(2)=7$	$f(3)=8$
case 7:	$f(1)=8$	$f(2)=8$	$f(3)=7$
case 8:	$f(1)=8$	$f(2)=8$	$f(3)=8$

(The chosen ordering of these 8 functions is *lexicographic* ('alphabetic') in terms of the outputs.)

Lexicographic (alphabetic) ordering

It is important to be able to systematically list things, being sure to neither leave anything out, nor list the same thing twice.

It is also useful to have a set arranged to enable **insertion** of **new** items, or **searches** to see whether some specified item is there or not.

A familiar example of systematic ordering is **alphabetical ordering** of English words: the set of strings

$$\{\texttt{cat, dog, wolf, aardvark}\}$$

is alphabetized (method?!) to

$$\{\texttt{aardvark, cat, dog, wolf}\}$$

To see that `lynx` is *not* on the list, we look down the list to see where it *would* be if it were there, and it's not there.

Coming from the other direction, to systematically list **all** 4-character strings made from letters `o`, `x` we might use a lexicographic ordering (`o` comes before `x`)

```
oooo
ooox
ooxo
ooxx
oxoo
oxox
oxxo
oxxx
xooo
xoox
xoxo
xoxx
xxoo
xxox
xxxo
xxxx
```

To systematically list **all** 2-character strings made from letters `a, b, c, d` use a lexicographic ordering

```
aa
ab
ac
ad
ba
bb
bc
bd
ca
cb
cc
cd
da
db
dc
dd
```

This is not mysterious, but when you order or list something you should tell *how*, and **lexicographic ordering** is common and useful.

A function $f : A \to B$ is **surjective** (=*onto*) if every element of the *target* set B is *hit* by some element of the *source* set A. That is, for every $b \in B$ there is $a \in A$ such that $f(a) = b$.

Example: the function $f : \{1, 2, 3\} \to \{4, 5\}$ given by

$$f(1) = 4 \quad f(2) = 4 \quad f(3) = 5$$

is surjective because both elements of the target are hit. But

$$f(1) = 4 \quad f(2) = 4 \quad f(3) = 4$$

is *not* surjective because the element 5 in the target is *missed*.

A function $f : A \to B$ is **injective** (=*one-to-one*) if every element of the *target* set B is hit by *at most* one element of the *source* set A. That is, for $a_1, a_2 \in A$ we have $f(a_1) = f(a_2)$ **only** when $a_1 = a_2$.

Example: the function $f : \{1, 2\} \to \{4, 5, 6\}$ given by

$$f(1) = 4 \quad f(2) = 6$$

is injective because no two elements of the source hit the same element of the target. But

$$f(1) = 4 \quad f(2) = 4$$

is *not* injective because the element 4 in the target is *hit twice*.

Counting without listing

For example, we can **count** the number of functions $f : A \to B$ from one set A to another set B **without listing** them.

To *refer* to the elements of A, *order* them. Suppose A has 4 elements and B has 7.

There are 7 possible outputs (in B) for the first input from A.

For each choice of output for first input, there are 7 possible outputs for *second* input from A.

For each choice of outputs for 1^{st} and 2^{nd} inputs, there're 7 possible outputs for 3^{rd} input.

For each choice of outputs for first, second, and third inputs, there are 7 choices for the output for the 4^{th} input.

Thus, altogether, there are

$$\underbrace{7 \times 7 \times 7 \times 7}_{4} = 7^4$$

functions from 4-element set to 7-element set.

Another Example

We can **count** the number of **injective** functions $f : A \to B$ from one set to another **without listing** them all. Again suppose A has 4 elements and B has 7.

There are 7 possible outputs (in B) for the first input from A.

For each choice of output for first input, there are $7 - 1$ possible outputs for the *second* input from A, since the output for the second input must be *different* from the output for the *first* input.

For each choice of outputs for first and second inputs, there are $7 - 2$ possible outputs for the *third* input from A, since it must be different from *both* the first and second outputs (which are not the same as each other).

And for each choice of outputs for first. second, and third inputs, there are $7 - 3$ choices for the output for the 4^{th} input since it must different from the first three outputs (which are all different).

Thus, altogether, there are

$$7 \times (7 - 1) \times (7 - 2) \times (7 - 3)$$

injective functions from a 4-element set to an 7-element set.

Count the number of **orderings** of a 4-element set. (without listing them).

There are 4 choices for the first element of the subset.

For each choice of first element there are $4-1$ remaining choices for second element, since we can't re-use the first choice.

For each choice of first and second elements there are $4-2$ choices for third element, since we can't re-use the first two (different) choices.

And just $4-3$ choices for the last element. So, altogether,

$$4 \times (4-1) \times (4-2) \times (4-3)$$

orderings of a 4-element set.

The **factorial** function and notation is convenient: for non-negative integer n

$$n! = n(n-1)(n-2)\ldots 4 \cdot 3 \cdot 2 \cdot 1$$

By convention

$$0! = 1$$

So the number of orderings of a set with n elements is $n!$

The **binomial coefficients** are

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = n \text{ choose } k$$

Note: the **definition** of the binomial coefficient does not promise anything about it. It is not immediately clear that it is an integer. (It *is*.) It is not immediately clear that it has anything to do with *choosing* anything, even though it is *pronounced* '`n choose k`'.

Count the 3-element subsets of a 7-element set, (without listing).

7 choices for the first element of the subset.

For each choice of first element there are $7 - 1$ choices for second element of the subset, since we can't re-use the first choice.

For each choice of first and second elements there are $7 - 2$ choices for third element of the subset, as we can't re-use the first two (different) choices.

But this approach imparts a fictitious **ordering** to the subset, and we must compensate.

We must **divide** by the number of possible orderings of 3 things, namely 3! from above.

Thus, the number of 3-element subsets of a 7-element set is

$$\frac{7(7-1)(7-2)}{3!} = \frac{7 \cdot 6 \cdot 5}{3!}$$

$$= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3! \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \binom{7}{3}$$

Count the number of *surjective* functions $f : \{1,2,3\} \to \{4,5\}$. This is inefficient but educational: Look at *all* functions to see *how* a function might fail to be surjective:

$$\begin{array}{llll}
\text{case 1:} & f(1)=4 & f(2)=4 & f(3)=4 \\
\text{case 2:} & f(1)=4 & f(2)=4 & f(3)=5 \\
\text{case 3:} & f(1)=4 & f(2)=5 & f(3)=4 \\
\text{case 4:} & f(1)=4 & f(2)=5 & f(3)=5 \\
\text{case 5:} & f(1)=5 & f(2)=4 & f(3)=4 \\
\text{case 6:} & f(1)=5 & f(2)=4 & f(3)=5 \\
\text{case 7:} & f(1)=5 & f(2)=5 & f(3)=4 \\
\text{case 8:} & f(1)=5 & f(2)=5 & f(3)=5
\end{array}$$

Looking down the list, only the first and last fail to miss one or the other of the two elements in the target set.

This can be thought about more systematically: To miss one or the other target element (cases 1, 8) all inputs go to a *single* output. There are two choices of the single output, so the number of surjections is

$$\text{(no. all) - (no. failures)} = 2^3 - 2$$

Example:

To count surjections

$$f : \{1, 2, 3, 4, 5\} \to \{8, 9\}$$

again the only failures are functions which have a single output for all inputs, since the target set has just two elements.

The number of surjections is thus

$$\text{(no. all) - (no. failures)} = 2^5 - 2$$

since (as above) the number of *all* functions from a 5-element set to 2-element set is 2^5.

Example:

To count surjections

$$f : \{1, 2, 3, 4, 5\} \to \{6, 7, 8, 9\}$$

take a different approach.

Since the target set has just one fewer than the source set, a surjective function can send just two inputs to the same output, and all others must go to different outputs.

So we count the number of 2-element subsets of $\{1, 2, 3, 4, 5\}$, and for each such choice there are $4(4 - 1)(4 - 2)(4 - 3)$ choices of outputs.

Thus, the number of surjections from 5-element to 4-element set is

$$\text{(no. 2-element subsets of source)} \times 4!$$

$$= \binom{5}{2} \times 4!$$

Example:

Without listing them, count the pairs of disjoint 3-element and 5-element subsets of a 12-element set.

There are 12 choices for the first element of the first set, $12-1$ for the second, $12-2$ for the third, so there are $12(12-1)(12-2)$ choices for an **ordered** subset of 3 elements. But this style of choosing artificially orders the chosen elements. To take this into account, divide by $3!$, the number of ways to order a set with 3 elements. (As earlier) there are

$$12(12-1)(12-2)/3! = \binom{12}{3}$$

choices for a 3-element subset of a 12-element set.

From the remaining $(12-3)$-element subset, there are $(12-3)$ choices for the first element of the second set, $(12-3)-1$ choices for the second element of the second set, and so on. Divide by $5!$ to discount the artificial ordering. So for each choice of the first set there are

$$(12-3)(12-3-1)\dots(12-3-5+1)/5!$$

Thus, altogether there are

$$\frac{12!}{(12-3)!\,3!}\cdot\frac{(12-3)!}{(12-3-5)!\,5!}$$

$$=\frac{12!}{(12-3-5)!\,3!\,5!}$$

choices of 3-element and 5-element subsets of a 12-element set.

Note that we get the same answer if the roles of 3 and 5 are reversed in the derivation.

One more counting problem

Count the number of sets of 3 disjoint 2-element subsets of a 12-element set.

As above, there are $\binom{12}{2}$ choices for the first (?!) subset, $\binom{12-2}{2}$ choices for the second, and $\binom{12-2-2}{2}$ choices for the third. *But* there is no ordering on the set of 2-element subsets, so our choice procedure will choose the same thing several times (unlike the case where the disjoint subsets are different sizes)! For example,

$$\{\{1,2\},\{7,8\},\{3,4\}\}$$

would be chosen separately as

$$\{\{7,8\},\{3,4\},\{1,2\}\}$$

and altogether 3! ways. Thus, we must divide by 3!, the number of ways to order 3 things, getting the final count

$$\binom{12}{2}\binom{12-2}{2}\binom{12-2-2}{2}/3!$$