
Review/Outline

Review:

Basic computations in finite fields

Reed-Solomon codes

Hamming codes

Bose-Chaudhuri-Hocquengham (BCH) codes

Testing for irreducibility

Frobenius automorphisms

Other roots of equations

Counting irreducibles

Counting primitive polynomials

New: Equation satisfied by field element

Counting irreducibles

[0.0.1] **Theorem:** Let p be prime. Fix a degree d . The number of irreducible **monic** (=leading coefficient 1) polynomials of degree d in $\mathbb{F}_p[x]$ is

$$\frac{p^d - \sum_{q|d} p^{d/q} + \sum_{q_1, q_2|d} p^{d/q_1 q_2} - \dots}{d}$$

where q_1, q_2, \dots are *distinct* primes dividing d .
For example, in some simple cases:
degree d prime:

$$\frac{p^d - p}{d}$$

$d = qr$ with q, r distinct primes:

$$\frac{p^{qr} - p^q - p^r + p}{qr}$$

$d = abc$ with a, b, c distinct primes:

$$\frac{p^{abc} - p^{ab} - p^{ac} - p^{bc} + p^a + p^b + p^c - p}{abc}$$

[0.0.2] **Remark:** The number of elements in the field, p , did not have to be prime, but only prime-power.

In more succinct (but possibly less transparent) notation: the number of irreducible polynomials of degree d in $\mathbb{F}_p[x]$ is

$$\frac{\sum_{i \geq 0} (-1)^i \sum_{i\text{-tuples } q_1 < \dots < q_i} p^{d/q_1 \dots q_i}}{d}$$

where for each i the i -tuples $q_1 < \dots < q_i$ runs over (distinct) primes dividing d .

Further examples:

for $d = q^2$ with q prime:

$$\frac{p^{q^2} - p^q}{q^2}$$

for $d = q^3$ with q prime:

$$\frac{p^{q^3} - p^{q^2}}{q^3}$$

for $d = q^4$ with q prime:

$$\frac{p^{q^4} - p^{q^3}}{q^3}$$

[0.0.3] Remark: It may not be entirely obvious that these are non-negative integer values.

[0.0.4] **Example:** Find the number of irreducible degree 2 polynomials with coefficients in \mathbb{F}_2 : The formula above, in the case of prime degree d , says that the number of irreducible degree d polynomials with coefficients in \mathbb{F}_p is

$$\frac{p^d - p}{d}$$

Here $p = 2$ and $d = 2$, so the number of such is

$$\frac{2^2 - 2}{2} = 1$$

Indeed, by trial and error we know it's $x^2 + x + 1$.

[0.0.5] **Example:** Number of irreducible degree 3 polynomials with coefficients in \mathbb{F}_2 : The formula above, in the case of prime degree d , says that the number of irreducible degree d polynomials with coefficients in \mathbb{F}_p is

$$\frac{p^d - p}{d}$$

Here $p = 2$ and $d = 3$, so the number is

$$\frac{2^3 - 2}{3} = 2$$

Indeed, by trial and error we know they're $x^3 + x + 1$ and $x^3 + x^2 + 1$.

[0.0.6] **Example:** Irreducible degree 4 polynomials with coefficients in \mathbb{F}_2 : The formula above, in the case of degree d the square of a prime q , says that the number of irreducible degree d polynomials with coefficients in \mathbb{F}_p is

$$\frac{p^{q^2} - p^q}{q^2}$$

Here $p = 2$ and $d = 4$, so $q = 2$, and the number of such is

$$\frac{2^4 - 2^2}{4} = 3$$

Again, by trial and error, we know they're

$$x^4 + x + 1$$

$$x^4 + x^3 + 1$$

and

$$x^4 + x^3 + x^2 + x + 1$$

[0.0.7] Example: Number of irreducible degree 5 polynomials with coefficients in \mathbb{F}_2 : The formula above, in the case of prime degree d , says that the number of irreducible degree d polynomials with coefficients in \mathbb{F}_p is

$$\frac{p^d - p}{d}$$

Here $p = 2$ and $d = 5$, so the number is

$$\frac{2^5 - 2}{5} = 6$$

[0.0.8] Example: Irreducible degree 6 polynomials with coefficients in \mathbb{F}_2 : The formula says the number of irreducible degree $d = qr$ polynomials with coefficients in \mathbb{F}_p , with distinct primes q, r , is

$$\frac{p^{qr} - p^q - p^r + p}{qr}$$

Here $p = 2$, $q = 2$, $r = 3$ so the number is

$$\frac{2^6 - 2^2 - 2^3 + 2}{6} = 9$$

[0.0.9] **Example:** Find the number of irreducible degree 2 polynomials with coefficients in \mathbb{F}_3 : The formula above, in the case of prime degree d , says that the number of irreducible degree d polynomials with coefficients in \mathbb{F}_p is

$$\frac{p^d - p}{d}$$

Here $p = 3$ and $d = 2$, so the number of such is

$$\frac{3^2 - 3}{2} = 3$$

Indeed, by trial and error they're

$$x^2 + 1$$

$$x^2 + x + 2$$

$$x^2 + 2x + 2$$

[0.0.10] **Example:** Number of irreducible degree 3 polynomials with coefficients in \mathbb{F}_3 : The formula above, in the case of prime degree d , says that the number of irreducible degree d polynomials with coefficients in \mathbb{F}_p is

$$\frac{p^d - p}{d}$$

Here $p = 3$ and $d = 3$, so the number is

$$\frac{3^3 - 3}{3} = 8$$

[0.0.11] **Example:** Irreducible degree 4 polynomials with coefficients in \mathbb{F}_3 : in the case of degree d the square of a prime q , the number of irreducible degree d polynomials with coefficients in \mathbb{F}_p is

$$\frac{p^{q^2} - p^q}{q^2}$$

Here $p = 3$ and $d = 4$, so $q = 2$, and the number is

$$\frac{3^4 - 3^2}{4} = 18$$

[0.0.12] **Example:** Find the number of irreducible degree 7 polynomials with coefficients in \mathbb{F}_3 : The formula above, in the case of prime degree d , says that the number of irreducible degree d polynomials with coefficients in \mathbb{F}_p is

$$\frac{p^d - p}{d}$$

Here $p = 3$ and $d = 7$, so the number of such is

$$\frac{3^7 - 3}{7} = 312$$

[0.0.13] **Example:** Find the number of irreducible degree 6 polynomials with coefficients in \mathbb{F}_3 : The formula above, in the case of degree $d = qr$ with primes q, r , says that the number of irreducible degree d polynomials with coefficients in \mathbb{F}_p is

$$\frac{p^{qr} - p^q - p^r + p}{qr}$$

Here $p = 3$ and $d = qr$ with $q = 2$ and $r = 3$, so the number of such is

$$\frac{3^6 - 3^3 - 3^2 + 3}{6} = 116$$

Euler's φ -function

A convenient counting function is Euler's φ -function, or **totient** function

$$\varphi(n) = \text{no. } t \text{ with } 1 \leq t \leq n \text{ and } \gcd(t, n) = 1$$

For example

$$\varphi(2) = \text{no.}\{1\} = 1$$

$$\varphi(3) = \text{no.}\{1, 2\} = 2$$

$$\varphi(4) = \text{no.}\{1, 3\} = 2$$

$$\varphi(5) = \text{no.}\{1, 2, 3, 4\} = 4$$

$$\varphi(6) = \text{no.}\{1, 5\} = 2$$

$$\varphi(7) = \text{no.}\{1, 2, 3, 4, 5, 6\} = 6$$

$$\varphi(8) = \text{no.}\{1, 3, 5, 7\} = 4$$

[0.0.14] **Remark:** Do *not* compute $\varphi(n)$ this way: do *not* enumerate the set to be counted.

[0.0.15] Theorem: Let

$$n = p_1^{e_1} \cdots p_t^{e_t}$$

be the factorization of n into primes, with $p_1 < \cdots < p_t$ and all $e_i \geq 1$. Then

$$\varphi(n) = (p_1 - 1)p_1^{e_1-1} \cdots (p_t - 1)p_t^{e_t-1}$$

[0.0.16] Examples:

$$\varphi(10) = \varphi(2 \cdot 5) = (2 - 1) \cdot (5 - 1) = 4$$

$$\varphi(12) = \varphi(2^2 \cdot 3) = (2 - 1)2 \cdot (3 - 1) = 4$$

$$\varphi(15) = \varphi(3 \cdot 5) = (3 - 1) \cdot (5 - 1) = 8$$

$$\varphi(30) = \varphi(2 \cdot 3 \cdot 5) = (2 - 1) \cdot (3 - 1) \cdot (5 - 1) = 8$$

$$\varphi(100) = \varphi(2^2 \cdot 5^2) = (2 - 1)2 \cdot (5 - 1)5 = 40$$

[0.0.17] Remark: This formula fails if we cannot factor n (presumably due to n being very large and having no small prime factors).

Counting primitives

[0.0.18] **Theorem:** Let q be a prime power. Fix a degree d . The number of *primitive monic* degree d polynomials in $\mathbb{F}_q[x]$ is

$$\frac{\varphi(q^d - 1)}{d}$$

[0.0.19] **Remark:** It is completely unclear that the given expression is an integer.

[0.0.20] **Remark:** We did not prove it, but it is true that *primitive* polynomials are necessarily *irreducible*: primitivity is a stronger condition than irreducibility.

[0.0.21] **Example:** To count primitive monic degree 2 polynomials in $\mathbb{F}_2[x]$, in the theorem $d = 2$ and $q = 2$, giving

$$\begin{aligned} \text{no. prim. quadratics} &= \frac{\varphi(2^2 - 1)}{2} \\ &= \frac{\varphi(3)}{2} = \frac{(3 - 1)}{2} = 1 \end{aligned}$$

using also the formula for Euler's φ -function.

[0.0.22] **Example:** To count primitive monic degree 3 polynomials in $\mathbb{F}_2[x]$, in the theorem $d = 3$ and $q = 2$, giving

$$\begin{aligned}\text{no. prim. cubics} &= \frac{\varphi(2^3 - 1)}{3} \\ &= \frac{\varphi(7)}{3} = \frac{(7 - 1)}{3} = 2\end{aligned}$$

using also the formula for Euler's φ -function (factoring by trial division). There are only two irreducible cubics $x^3 + x + 1$ and $x^3 + x^2 + 1$ so they are necessarily primitive.

[0.0.23] **Example:** To count primitive monic degree 4 polynomials in $\mathbb{F}_2[x]$, in the theorem $d = 4$ and $q = 2$, giving

$$\begin{aligned}\text{no. prim. quartics} &= \frac{\varphi(2^4 - 1)}{4} \\ &= \frac{\varphi(15)}{4} = \frac{(3 - 1)(5 - 1)}{4} = 2\end{aligned}$$

using also the formula for Euler's φ -function (factoring by trial division). Of the 3 irreducible quartics $x^4 + x + 1$ and $x^4 + x^3 + 1$ are primitive.

[0.0.24] **Example:** To count primitive monic degree 5 polynomials in $\mathbb{F}_2[x]$, in the theorem $d = 5$ and $q = 2$, giving

$$\begin{aligned} \text{no. prim. quintics} &= \frac{\varphi(2^5 - 1)}{5} \\ &= \frac{\varphi(31)}{5} = \frac{(31 - 1)}{5} = 6 \end{aligned}$$

using also the formula for Euler's φ -function (31 is prime, by trial division). There are $(2^5 - 2)/5 = 6$ irreducibles, to every irreducible is primitive. Indeed, generally,

[0.0.25] **Corollary:** If $q^d - 1$ is prime, then every irreducible monic degree d polynomial in $\mathbb{F}_q[x]$ is primitive (otherwise, there will be fewer primitives than irreducibles of a given degree).

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[0.0.26] Example: To count primitive monic degree 6 polynomials in $\mathbb{F}_2[x]$, in the theorem $d = 6$ and $q = 2$, giving

$$\begin{aligned} \text{no. prim. sextics} &= \frac{\varphi(2^6 - 1)}{6} \\ &= \frac{\varphi(63)}{6} = \frac{(3 - 1)3 \cdot (7 - 1)}{6} = 6 \end{aligned}$$

using the formula for Euler's φ -function, factoring 63 by trial division.

[0.0.27] Example: To count primitive monic degree 7 polynomials in $\mathbb{F}_2[x]$, in the theorem $d = 7$ and $q = 2$, giving

$$\begin{aligned} \text{no. prim. sextics} &= \frac{\varphi(2^7 - 1)}{6} \\ &= \frac{\varphi(127)}{7} = \frac{(127 - 1)}{7} = 18 \end{aligned}$$

using the formula for Euler's φ -function (127 is prime by trial division).

[0.0.28] **Example:** To count primitive monic degree 2 polynomials in $\mathbb{F}_3[x]$, in the theorem $d = 2$ and $q = 3$, giving

$$\begin{aligned} \text{no. prim. sextics} &= \frac{\varphi(3^2 - 1)}{2} \\ &= \frac{\varphi(8)}{2} = \frac{(2 - 1)2^{3-1}}{2} = 2 \end{aligned}$$

using the formula for Euler's φ -function, factoring 8 by trial division.

Among the 3 irreducible quadratics $x^2 + 1$, $x^2 + x + 2$, $x^2 + 2$ here, which is the non-primitive one? For $P(x) = x^2 + 2$, note that

$$x^2 = (x^2 + 2) + 1 = 1 \pmod{P(x)}$$

so x has *order* just 2 mod $P(x)$, not the maximal possible, namely $3^2 - 1 = 8$. So $x^2 + 2$ is the non-primitive one.

[0.0.29] **Example:** To count primitive monic degree 3 polynomials in $\mathbb{F}_3[x]$, in the theorem $d = 3$ and $q = 3$, giving

$$\begin{aligned} \text{no. prim. cubics} &= \frac{\varphi(3^3 - 1)}{3} \\ &= \frac{\varphi(26)}{3} = \frac{(2 - 1) \cdot (13 - 1)}{3} = 4 \end{aligned}$$

using the formula for Euler's φ -function (factoring by trial division).

[0.0.30] **Remark:** If we believe the formula, then there really *are* primitive polynomials of all degrees.