quiz 02.1 Solution

(1) You flipped a fair coin 71 times in a row, out of which there were 28 heads and 43 tails. What is the probability that the next flip will be a head?

We assume that different coin flips are independent, so the probabilities of the outcomes of the next flip do not depend at all upon the previous outcomes. Since the coin is assumed 'fair' that means that $P(T) = P(H)$, so (since the sum is 1) $P(H) = P(T) = 1/2$, and thus the probability will be '1/2'.

(2) There are 4 blue balls and 3 red balls in an urn. What is the probability of drawing at least 6 blue balls out of 9 draws (with replacement)?

Our usual assumptions lead us to believe that the probability of drawing a blue ball in a single draw is $4/7$, since the total number of balls is $7 = 4 + 3$, and also to posit that the different draws are *independent*. To draw at least 6 blue balls means to draw *exactly* either $6, 6 + 1, 6 + 2, \ldots$. The number of ways to draw exactly ℓ blue balls in 9 draws is equal to the number of ways of choosing ℓ things from 9, which by now we know is a binomial coefficient $\binom{9}{\ell}$. The probability of drawing exactly ℓ blue balls is $\binom{9}{\ell}(\frac{4}{7})^\ell(\frac{3}{7})^{9-\ell}$. Adding these up, the desired probability is

$$\binom{9}{6}(\frac{4}{7})^6(\frac{3}{7})^{9-6} + \ldots + \binom{9}{9}(\frac{4}{7})^9(\frac{3}{7})^{9-9} \approx 0.412100955436276$$

(3) There are 3 red balls and 4 blue balls in an urn. What is the expected number of red balls drawn in 13 draws (with replacement)?

Let X be the random variable which counts the number of red balls in 13 draws. Then

$$X = X_1 + X_2 + \ldots + X_{13}$$

where X_i is the random variable that tells the number of red balls on the i-th draw. We will invoke the little theorem that asserts that expected values are *linear*, namely

$$EX = \sum_{i=1}^{13} EX_i$$

By the assumed independence of the different draws, it is easy to evaluate the expected values of the X_i's directly from the definition of expected value:

$$EX_i = \sum_{\text{values } x \text{ of } X_i} x \cdot P(X_i = x) = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = 1 \cdot \frac{3}{3+4} + 0 \cdot \frac{4}{3+4} = \frac{3}{3+4}$$

(As usual we assume that drawing any ball has the same probability as any other, from which we conclude that the probabilities are as indicated.) Then

$$EX = \sum_{i=1}^{13} EX_i = \sum_{i=1}^{13} \frac{3}{3+4} = 13 \cdot \frac{3}{3+4}$$

(4) There are 7 red balls and 12 blue balls in an urn. You draw repeatedly (with replacement). What is the expected number of draws necessary so that you'll have drawn at least one ball of each color?

Since the colored balls are assumed otherwise indistinguishable, the probability of drawing a red ball is $p = 7/(7 + 12)$ and the probability of drawing a blue ball is $q = 12/(7 + 12)$. These are non-negative, with sum equal to 1. The expected value is

$$\sum_{\ell=2}^{\infty} \ell \cdot P(\text{one color for } \ell - 1, \text{ then get the second color on } \ell\text{th })$$

(The summation starts at 2 since it will take at least 2 draws to get 2 different colors.) This is

$$\sum_{\ell \geq 2} \ell \cdot [P(\ell - 1 \text{ blue, then } 1 \text{ red}) + P(\ell - 1 \text{ red, then } 1 \text{ blue})]$$

Note that there is no overlap among the cases: either we get all blues for a while and then a red, or vice-versa, and we stop as soon as we get the missing color. Then the expected value is

$$\sum_{\ell \geq 2} \ell \cdot [q^{\ell-1} \cdot p + p^{\ell-1} \cdot q]$$

The real problem here is that we have indicated an infinite sum, which is certainly not acceptable. We differentiate a geometric series summation to obtain a handy identity:

$$\sum_{\ell \geq 2} x^\ell = x^2/(1-x) = ((x^2-1)+1)/(1-x) = -x - 1 + \frac{1}{1-x}$$

$$\sum_{\ell \geq 2} \ell \cdot x^{\ell-1} = \frac{\partial}{\partial x} \sum_{\ell \geq 2} x^\ell = \frac{\partial}{\partial x} \left(-x - 1 + \frac{1}{1-x} \right) = -1 + \frac{1}{(1-x)^2}$$

Applying this and keeping in mind that $p + q = 1$, we have

$$\sum_{\ell \geq 2} [q^{\ell-1} \cdot p + p^{\ell-1} \cdot q] = [-1 + \frac{1}{(1-q)^2}] \cdot p + [-1 + \frac{1}{(1-p)^2}] \cdot q = -(p+q) + \frac{1}{p^2} \cdot p + \frac{1}{q^2} \cdot q = -1 + \frac{1}{p} + \frac{1}{q}$$

Substituting the numerical values 7/19 for p and 12/19 for q in the latter formula gives an expected wait of $277/84 \approx 3.29761904761905$ draws to see both colors.