quiz 03.1 Solution

(1) Given that the roll of a pair of fair dice yields a sum at least 5, what is the probability that the sum is exactly 7?

Use the definition of conditional probability, namely that for any two events A and B

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Apply this here to A being the event that the sum is 7, and B being the event that the sum is at least 5. In the present case $A \subset B$, so here

$$P(A|B) = \frac{P(A)}{P(B)}$$

The assumption that the dice are fair means that every one of the $6 \times 6 = 36$ possible ordered pairs of face values is equally likely, thus having probability 1/36. Thus, the probability of getting a value s as sum is 1/36 times the number of ways of getting s as the sum of two integers in the range from 1 to 6. That is,

$$\begin{array}{ccc}
2 = 1+1 & \text{so} & P(\text{sum is } 2) = 1/36 \\
3 = 1+2 = 2+1 & \text{so} & P(\text{sum is } 3) = 2/36 \\
4 = 1+3 = 2+2 = 3+1 & \text{so} & P(\text{sum is } 4) = 3/36 \\
 & \ldots & \\
10 = 4+6 = 5+5 = 6+4 & \text{so} & P(\text{sum is } 10) = 3/36 \\
11 = 5+6 = 5+6 & \text{so} & P(\text{sum is } 11) = 2/36 \\
12 = 6+6 & \text{so} & P(\text{sum is } 12) = 1/36
\end{array}$$

Then the conditional probability of getting 7 given that the sum is at least 5 is

$$P(7| \geq 5) = \frac{P(7)}{P(\geq 5)} = \frac{P(7)}{P(5) + \ldots + P(12)}$$

because the various values are disjoint events. Numerically, this is

$$\frac{6/36}{4/36 + \ldots + 1/36} = \frac{6}{30} \approx 0.2$$

(2) You have a coin which is not fair: it gives Heads with probability 1/4 and Tails with probability 3/4. What is the entropy of the random variable which counts the number of heads in 5 flips of this coin?

Let X be this random variable. Use notation

$$H(X) = -\sum_x P(X = x) \log_2 P(X = x)$$

where x is summed over the values that x takes.

$$H(X) = -\sum_{x=0,\ldots,5} P(X = x) \log_2 P(X = x)$$

By now we are very well acquainted with facts such as the assertion that the probability of ℓ H's in 5 draws is $\binom{5}{\ell} 0.25^{\ell} 0.75^{5-\ell}$. Then, wisely abbreviating $h(p) = -p \log_2 p$, the entropy is

$$H(X) = h((1/4)^5) + h\left(\binom{5}{1}(1/4)^{5-1}(3/4)^1\right) + \ldots + h\left(\binom{5}{5-1}(1/4)^1(3/4)^{5-1}\right) + h\left((3/4)^5\right)$$

$$\approx 1.93617195626818$$

(3) Find all possible decodings of the message 10010110101011010 which was encoded by the scheme $a = 10$, $b = 100$, $c = 10101$, $d = 10110$.

We'll use the simple method of decoding which starts from the left and tries to decode (in all possible ways) character by character, considering all possibilities, and backtracking both if failure is encountered after partially successful decoding and to find all other possible decodings after a successful complete decoding is found. Partial decoding: 'b' leaves '10110101011010'. Partial decoding: 'bd' leaves '101011010'. Partial decoding: 'bdc' leaves '1010'. Partial decoding: 'bdca' leaves '10'. Then one complete decoding is 'bdcaa'. To find more possible decodings, backtrack to last decoding where there was any alternative decoding. Partial decoding: 'bda' leaves '1011010'. Partial decoding: 'bdad' leaves '10'. Then one complete decoding is 'bdada'. To find more possible decodings, backtrack to last decoding where there was any alternative decoding. Partial decoding: 'bdaa' leaves '11010'. Partial decoding 'bdaa' cannot be continued. To find more possible decodings, backtrack to last decoding where there was any alternative decoding. Partial decoding: 'ba' leaves '110101011010'. Partial decoding 'ba' cannot be continued. To find more possible decodings, backtrack to last decoding where there was any alternative decoding. Partial decoding: 'a' leaves '010110101011010'. Partial decoding 'a' cannot be continued. To find more possible decodings, backtrack to last decoding where there was any alternative decoding. No more possibilities remain. Thus, the list of all possible decodings is 'bdcaa', 'bdada'.

(4) Does there exist a uniquely-decodable binary code with word lengths 2, 2, 3, 3, 3, 3, 5, 5?

The Kraft-MacMillan theorem addresses this exactly: there exists such a code if and only if

$$\sum_{\text{lengths } \ell \text{ of words}} \frac{1}{2^\ell} \leq 1$$

The powers of 2 occurring in the denominator in this example are 4, 4, 8, 8, 8, 8, 32, 32, and the sum of their inverses is

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{32} + \frac{1}{32} = 1.0625$$

Since this sum is greater than 1, there is no such code.