quiz 08.1 Solution

(1) Find a linear dependence relation among the vectors 1000, 1110, 0011, 1111, 0110.

Stack these vectors into a 5-by-4 matrix, augment it by sticking a 5-by-5 identity matrix on the right, and do row reduction. When you're done, the bottom row of what was originally the 5-by-5 identity matrix will be the coefficients of the desired linear relation. That is: The matrix is

$$\begin{pmatrix} 1&0&0&0&1&0&0&0&0\\ 1&1&1&0&0&1&0&0&0\\ 0&0&1&1&0&0&1&0&0\\ 1&1&1&1&0&0&0&1&0\\ 0&1&1&0&0&0&0&0&1 \end{pmatrix}$$

Add row 0 to row 1, to obtain

$$\begin{pmatrix} 1&0&0&0&1&0&0&0&0\\ 0&1&1&0&1&1&0&0&0\\ 0&0&1&1&0&0&1&0&0\\ 1&1&1&1&0&0&0&1&0\\ 0&1&1&0&0&0&0&0&1 \end{pmatrix}$$

Add row 0 to row 3, to obtain

$$\begin{pmatrix} 1&0&0&0&1&0&0&0&0\\ 0&1&1&0&1&1&0&0&0\\ 0&0&1&1&0&0&1&0&0\\ 0&1&1&1&1&0&0&1&0\\ 0&1&1&0&0&0&0&0&1 \end{pmatrix}$$

Add row 1 to row 3, to obtain

$$\begin{pmatrix} 1&0&0&0&1&0&0&0&0\\ 0&1&1&0&1&1&0&0&0\\ 0&0&1&1&0&0&1&0&0\\ 0&0&0&1&0&1&0&1&0\\ 0&1&1&0&0&0&0&0&1 \end{pmatrix}$$

Add row 1 to row 4, to obtain

$$\begin{pmatrix} 1&0&0&0&1&0&0&0&0\\ 0&1&1&0&1&1&0&0&0\\ 0&0&1&1&0&0&1&0&0\\ 0&0&0&1&0&1&0&1&0\\ 0&0&0&0&1&1&0&0&1 \end{pmatrix}$$

Thus, reading from the bottom row of the right half of this reduced matrix, the coefficients of the linear dependence relation are 1, 1, 0, 0, 1:

$$1 \cdot 1000 + 1 \cdot 1110 + 0 \cdot 0011 + 0 \cdot 1111 + 1 \cdot 0110 = 0$$

is the desired linear dependence relation among the given vectors.

(2) What is the dimension of the vector subspace of $\mathbf{F}_2^6$ spanned by vectors 110111, 110011, 000001, 100100, 000011, 100010, 100110?

Stack the vectors into a 7-by-6 matrix, and row reduce. Note that row reduction does not change the row space of a matrix. When you're done, the remaining non-zero rows are a basis for the row space of the original matrix. The matrix is

$$\begin{pmatrix} 1&1&0&1&1&1\\ 1&1&0&0&1&1\\ 0&0&0&0&0&1\\ 1&0&0&1&0&0\\ 0&0&0&0&1&1\\ 1&0&0&0&1&0\\ 1&0&0&1&1&0 \end{pmatrix}$$

Add row 0 to row 1, to obtain

$$\begin{pmatrix} 1&1&0&1&1&1\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1\\ 1&0&0&1&0&0\\ 0&0&0&0&1&1\\ 1&0&0&0&1&0\\ 1&0&0&1&1&0 \end{pmatrix}$$

Add row 0 to row 3, to obtain

$$\begin{pmatrix} 1&1&0&1&1&1\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1\\ 0&1&0&0&1&1\\ 0&0&0&0&1&1\\ 1&0&0&0&1&0\\ 1&0&0&1&1&0 \end{pmatrix}$$

Add row 0 to row 5, to obtain

$$\begin{pmatrix} 1&1&0&1&1&1\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1\\ 0&1&0&0&1&1\\ 0&0&0&0&1&1\\ 0&1&0&1&0&1\\ 1&0&0&1&1&0 \end{pmatrix}$$

Add row 0 to row 6, to obtain

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Interchange 3 and 1 rows, to get

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Add row 1 to row 5, to obtain

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Add row 1 to row 6, to obtain

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Interchange 3 and 2 rows, to get

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Add row 2 to row 5, to obtain

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Interchange 4 and 3 rows, to get

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Add row 3 to row 5, to obtain

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Add row 3 to row 6, to obtain

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Add row 4 to row 5, to obtain

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Add row 4 to row 6, to obtain

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The last matrix is the row reduced form of the original matrix, and its non-zero rows are a basis for the row space. Thus, the dimension is 5.

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(3) Find a check matrix H for the binary code C with generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

One choice of check matrix H for a k-by-n generating matrix G of the special form $G = (1_k \quad A)$ is obtained as $H = (-A^t \quad 1_{n-k})$. Thus, in the case at hand, H is

$$H = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

(4) Let C be the cyclic binary code of length 9 with generator polynomial 100111011 (coefficients ordered by ascending degree). Find a check matrix H for C.

The check polynomial $h(x)$ is obtained by **reversing coefficients** in

$$\frac{x^{\text{length}} - 1}{\gcd(x^{\text{length}} - 1, v)} = \frac{x^9 - 1}{\gcd(x^9 - 1, v)}$$

where we view v as giving a the coefficients of a polynomial, in ascending order. Computing the gcd by the Euclidean algorithm we find that the gcd is (as binary vector) 1001. Thus, $h(x)$ is the coefficients-reversed version of the quotient $(x^9 - 1)/(x^3 - 1) = x^6 + x^3 + 1$, which is the same thing back again, and the check matrix is

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Note that we cycle the row of the check matrix until it bumps into the right-hand side.

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