I. Automorphic representations and Galois representations

I will be lecturing on joint work in progress with Richard Taylor, and in part with Laurent Clozel. Our goal is to prove that certain kinds of finite-dimensional representations ρ of $G_E = Gal(\bar{E}/E)$, where E is a number field, are associated to automorphic forms. My goal in the first talk of the series is to make the goal of our work in progress comprehensible. This means, in particular, addressing the following questions:

1. What kinds of representations of G_E do we have in mind?
2. What properties do the representations have to have in order to be associated to automorphic forms, and for what E is this conceivable?
3. What kinds of automorphic forms have representations associated to them, and how?
4. What does "associated" mean, anyway?

A fifth question, namely why one would want to look at representations of G_E at all, is addressed in my Junior Colloquium talk.

Here are the short answers:

1. Ans. ρ is an ℓ-adic representation, i.e. an n-dimensional representation on a vector space over a finite extension of $\mathbb{Q}_\ell$, where ℓ is some prime number.
2. Ans. For the time being, E has to be totally real or a CM field, and ρ has to be of *geometric type* in the terminology of Fontaine-Mazur. I will try to explain this slowly. This is the full scope of the Langlands conjectures. For the foreseeable future, one has to make additional restrictive assumptions on the Galois representations.
3. Ans. The model is always the theorem of Eichler-Shimura, which attaches two-dimensional ℓ-adic representations to holomorphic modular forms of weight 2, and its generalization by Deligne to weights $k \geq 2$. For more general Galois representations one has to work with automorphic representations π of $GL(n, E)$ – I will say something about what this means – rather than individual automorphic forms. The class of π which one can hope to associate to Galois representations was identified in Clozel's Ann Arbor article; they are those of *algebraic type*. Parallel to the additional restrictions on the ρ are additional restrictions on the π; these are the natural generalizations of modular forms of weight $k \geq 2$, but recognizing them as "natural" takes some practice.
4. Ans. Both ρ and the automorphic representation π_ρ can be identified from their L-functions $L(s, \rho)$, $L(s, \pi_\rho)$ which are Dirichlet series with Euler products. They are associated if the L-functions are equal. The theory of automorphic forms then shows that $L(s, \pi_\rho)$ has an analytic (or at least meromorphic) continuation to all $s \in \mathbb{C}$ satisfying a functional equation. Thus the assertion that the two are associated implies that the same is true of $L(s, \rho)$, and this is indeed the only way one knows to prove analytic continuation of the latter.

I will begin with automorphic representations. Henceforth, F will always be a CM field, $c \in Aut(F)$ the (unique) complex conjugation, $F^+ \subset F$ the fixed field of F, which is totally real. The letter E denotes a number field which is either F or F^+.

1. Automorphic representations of $GL(n, E)$.

A holomorphic modular form F of weight k is a holomorphic function on the upper half-plane $\mathfrak{H}$ satisfying a certain symmetry. The group $SL(2,\mathbb{R})$ acts by linear-fractional transformations on $\mathfrak{H}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}$$

and F is assumed to transform according to a certain rule, depending on k, for $\gamma \in \Gamma$ where $\Gamma \subset SL(2,\mathbb{Q}) \subset SL(2,\mathbb{R})$ is a congruence subgroup. There is also a growth condition near the boundary of $\mathfrak{H}$. I assume this is all familiar and I mention this in a largely futile attempt to motivate the definition of automorphic forms in general. With little effort one extends F to a function on $\mathfrak{H} \coprod \bar{\mathfrak{H}}$ (upper + lower halfplanes), which is identified with $GL(2,\mathbb{R})/SO(2) \cdot \mathbb{R}^{\times}$. A standard lifting procedure lifts F to a function f on $\Gamma \backslash GL(2,\mathbb{R})$, and thence to a function

$$f : GL(2,\mathbb{Q}) \backslash GL(2,\mathbf{A}) \to \mathbb{C}$$

where in each case the transformation rule itself undergoes a transformation. The original holomorphy condition is inherited by f in another form. For purposes of generalization, it is best to think of a holomorphic form of weight 2 as a holomorphic 1-form on the open complex curve $\Gamma\backslash\mathfrak{H}$, and then just as a cohomology class, possibly with some funny behavior at the missing points. Forms of weight $k > 2$ define cohomology with twisted coefficients. That is what generalizes.

Following earlier work of Mordell, Hecke discovered the operators that bear his name on the spaces of modular forms, and more importantly, that the simultaneous eigenfunctions for all these operators could be assigned Dirichlet series with analytic continuation and functional equations. The extra adelic variables allow a natural definition of the Hecke operators. A (discrete) *automorphic representation* of $GL(2,\mathbb{Q})$ is a direct summand of

$$L^2(GL(2,\mathbb{Q}) \backslash GL(2,\mathbf{A})/\mathbb{R}^{\times})$$

which defines an irreducible representation of $GL(2,\mathbf{A})$ under the right regular representation. When one applies the lifting procedure of the previous paragraph to one of the eigenforms Hecke considered, its right $GL(2,\mathbf{A})$-translates form an automorphic representation in this sense, and all the eigenvalues of the Hecke operators can be recovered from the structure of the abstract representation. Most discrete automorphic representations are *cuspidal*, which means they decrease rapidly near infinity, and we only consider cuspidal representations.

Now we can define an automorphic representation π of $GL(n,E)$ to be a direct summand of

$$L^2(GL(n,E) \backslash GL(n,\mathbf{A}_E)/(E \otimes_{\mathbb{Q}} \mathbb{R})^{\times}).$$

This is not a great definition, because of the quotient on the right, but I won't dwell on that. What's important is that it is an irreducible representation of $GL(n,\mathbf{A}_E)$, and that it consists of functions on the latter which are left-invariant under $GL(n,E)$. This left-invariance translates into a reciprocity condition that is used to define the analytic continuation of the L-function, and to motivate the hope of a connection to Galois representations. Here is what you need to know about automorphic representations π:

(1) Let $\{v\}$ denote the set of places of E, partitioned among finite (non-archimedean) and infinite (real and complex) places. Thus $GL(n, \mathbf{A}_E) = \prod'_v GL(n, E_v)$ (restricted direct product). Any irreducible representation π of $GL(n, \mathbf{A}_E)$ is isomorphic to a (restricted) tensor product $\otimes'_v \pi_v$, where π_v is an irreducible admissible representation of $GL(n, E_v)$.

(2) For almost all finite v, π_v has a non-zero fixed vector φ_v under $GL(n, \mathcal{O}_v)$ (spherical), which is then necessarily unique up to scalar multiples. For such representations Hecke operators can be defined. The set of spherical representations is in bijection (Satake isomorphism) with unordered n-tuples of non-zero complex numbers

$$\pi_v \leftrightarrow \{\alpha_{1,v}, \ldots, \alpha_{n,v}\}$$

and we can define a local Euler factor

$$L(s, \pi_v) = \prod_{i=1}^{n} (1 - \alpha_{i,v} N v^{-s})^{-1}.$$

Incidentally, this explains the notion of "restricted tensor product": $\otimes'_v \pi_v$ is the subspace of the abstract tensor product $\otimes_v \pi_v$ spanned by vectors of the form $\otimes_v \psi_v$, where for almost all finite v ψ_v is the chosen spherical vector φ_v.

The spherical representation π_v can be constructed as follows. Let $B \subset GL(n)$ be the upper-triangular Borel subgroup, $B(E_v) = AN$, with $A = E_v^{\times,n}$ the diagonal subgroup, with maximal compact subgroup $A^o = \mathcal{O}_v^{\times,n}$ The n-tuple $\{\alpha_{1,v}, \ldots, \alpha_{n,v}\}$ defines a character $\chi : A/A^o \to \mathbb{C}^\times$: we identify $A/A^o = (E_v^\times / \mathcal{O}_v^\times)^n \xrightarrow{\sim} \mathbb{Z}^n$ and send the i-th generator e_i of $\mathbb{Z}^n$ (an orientation is provided by the absolute value) to $\alpha_{i,v}$. Then χ lifts to a homomorphism $B \to \mathbb{C}^\times$ and in most cases $\pi_v = Ind_{B(E_v)}^{GL(n,E_v)} \chi$, where the induction is normalized to make this independent of the order. In all cases π_v is the unique spherical subquotient of the induced representation. More generally, representations of the form $Ind_{B(E_v)}^{GL(n,E_v)} \chi$ are called *unramified principal series*; they have finite composition series.

(3) More generally, if $n = \sum_{i=1}^r n_i$ is a partition, with $r > 1$, let $P \subset GL(n)$ be the parabolic subgroup with Levi factor $\prod_i GL(n_i)$. To any r-tuple of irreducible admissible representations σ_i of $GL(n_i, E_v)$ one can define $Ind_{P(E_v)}^{GL(n,E_v)} \sigma_1 \otimes \cdots \otimes \sigma_r$. This induced representation has finite length as a representation of $GL(n, E_v)$. The irreducible representations of $GL(n, E_v)$ that do not occur in any such composition series are called supercuspidal. They have the property of occurring as direct summands in $L^2(GL(n, E_v))$ (one has to make allowances for the action of the center of $GL(n, E_v)$ for this to be exactly right). Supercuspidal representations do not exist for archimedean v but they do exist for all finite v.

Irreducible admissible representations of $GL(n, E_v)$ are classified by the *local Langlands correspondence* with n-dimensional representations of the

Weil-Deligne group WD_{E_v} of E_v. We write $\mathcal{L}(\pi_v)$ for the corresponding n-dimensional representation of WD_{E_v}. If $n = \sum n_i$ as above and σ_i is an irreducible admissible representation of $GL(n_i, E_v)$, we write $\boxplus_i \sigma_i$ for the representation of $GL(n, E_v)$ such that

$$\mathcal{L}(\boxplus_i \sigma_i) = \oplus_i \mathcal{L}(\sigma_i).$$

Then $\boxplus_i \sigma_i$ is an irreducible constituent of $Ind_{P(E_v)}^{GL(n,E_v)} \sigma_1 \otimes \cdots \otimes \sigma_r$.

We also need to consider the Steinberg representations $St(n, \chi)$, where χ is a character of $E_v^\times$ (not necessarily unramified). These can be defined in various ways; the simplest is to consider that one-dimensional spherical representations of $GL(n, E_v)$ can be written uniquely as quotients of certain induced representations from $B(E_v)$. The Steinberg representations are the unique irreducible subrepresentations (not subquotients!) of these induced representations. The Steinberg representations are the only constituents of principal series that belong to the discrete series (i.e., that occur as direct summands in $L^2(GL(n, E_v))$).

(4) For v archimedean, one really wants to look at the differentiable vectors in π_v; these form a module for the enveloping algebra $U(\mathfrak{g}_v)$ of $\mathfrak{g}_v = Lie(GL(n, E_v))$. One restricts further to the K_v-finite vectors, where $K_v \subset GL(n, E_v)$ is a maximal connected compact subgroup: either $K_v = SO(n)$ for v real or $K_v = U(n)$ for v complex. It's difficult to say anything precise about these representations without developing the whole of representation theory of reductive Lie groups, but two classes of representations stand out. First, there are the irreducible finite-dimensional representations, characterized by their highest weights. These never occur as components of cuspidal automorphic representations. However, to certain finite-dimensional (τ, W) of $GL(n, E_v)$ one can associate representations $\pi_{v,\tau}$ of $GL(n, E_v)$ – or rather $(\mathfrak{g}_v, K_v)$-modules, since they are actually algebraic objects – which do occur in cuspidal automorphic representations. We can characterize $\pi_{v,\tau}$ as a representation on which the center of the enveloping algebra $U(\mathfrak{g}_v)$ acts by the same character as on the dual $\tau^\vee$ of τ, and which is also *generic* in the sense of having a Whittaker model. Alternatively, $\pi_{v,\tau}$ is the unique generic representation for which the relative Lie algebra cohomology

$$H^*(\mathfrak{g}_v, K_v; \pi_{v,\tau} \otimes W) \neq 0.$$

Without defining this functor, it suffices to know that this means precisely that automorphic representations with $\pi_{v,\tau}$ as local component contribute to the cohomology of $\Gamma \backslash GL(n, E_v)/K_v \cdot E_v^\times$ with twisted coefficients (modeled on τ), just as holomorphic modular forms contribute to cohomology of modular curves with twisted coefficients.

The generic representations $\pi_{v,\tau}$ are called *cohomological*. They don't exist for all (τ, W). For us, it suffices to consider τ satisfying either $\tau^\vee \xrightarrow{\sim} \tau$ (real places) or $\tau^\vee \xrightarrow{\sim} \tau^c$ (complex places). Then $\pi_{v,\tau}$ is also either isomorphic to its dual (for v real) or to its conjugate-dual (for v complex). The trivial representation is a special case that will suffice for most illustrations. The corresponding $\pi_{v,\tau}$ are components of representations on the usual cohomology of $\Gamma \backslash GL(n, E_v)/K_v \cdot E_v^\times$.

(5) The non-spherical π_v have Euler factors $L(s,\pi_v)$, which are essentially shifted products of Γ functions for archimedean v. There are also local constants $\varepsilon(s,\pi_v,\psi_v)$, where ψ_v is an additional datum that disappears in the product: $\varepsilon(s,\pi) = \prod_v \varepsilon(s,\pi_v,\psi_v)$. For π cuspidal, the Euler product $\Lambda(s,\pi) = \prod_v L(s,\pi_v)$ is entire and satisfies a functional equation:

$$\Lambda(s,\pi) = \varepsilon(s,\pi)\Lambda(1-s,\pi^\vee).$$

(We follow tradition and write $L(s,\pi) = \prod_{v\ finite} L(s,\pi_v)$.)

(6) We will make use of the property of cyclic base change, proved by Arthur and Clozel. Let E'/E be a finite cyclic Galois extension, and let π be an automorphic representation of $GL(n,E)$ as above. Then there exists an automorphic representation $\pi_{E'}$ of $GL(n,E')$. Note that $\pi_{E'}$ is not always necessarily cuspidal or discrete, even if π is, so something more needs to be said; in practice, we will arrange that $\pi_{E'}$ is cuspidal. One can put together $\pi_{E'}$ from its local components. If v splits completely in E' then for each v' dividing v, $\pi_{E',v'} \simeq \pi_v$. If v is inert in E', v' its unique divisor, then $\pi_{E',v'}$ is the local base change of π_v, which is defined by a character identity. The general situation is intermediate between these two extremes.

The group $Gal(E'/E)$ acts on the local components of $\pi_{E'}$ and fixes them by construction. More generally, if π' is an automorphic representation of $GL(n,E')$ fixed by $Gal(E'/E)$, then it equals the base change of an automorphic representation of $GL(n,E)$. (This is not strictly speaking a theorem unless $[E':E]$ is of prime degree, but it will always be the case in the situations we consider.) In the next section we will have two actions for the quadratic extension F/F^+. Suppose Π is an automorphic representation of $GL(n,F)$. If $\Pi \xrightarrow{\sim} \Pi^c$, we have already seen that Π is a base change from $GL(n,F^+)$. But if $\Pi \xrightarrow{\sim} (\Pi^c)^\vee$, Π will be a twisted base change from certain kinds of unitary groups.

(7) Pursuing the theme of base change, we have the following theorem:

Theorem. *Let π be a cuspidal automorphic representation of $GL(n,E)$. There exists a finite sequence of extensions $E_N \supset \dots E_i \supset E_{i-1} \supset \dots E_0 = E$, with E_i/E_{i-1} cyclic for all i, such that the result of successive base changes of π to E_N is cuspidal and has the property that, for all finite v, $\pi_{E_N,v}$ is a constituent of an unramified principal series representation. If E is CM (totally real) then all E_i can be taken to be CM (totally real) as well.*

This is a simple consequence of the local Langlands correspondence, base change, and weak approximation. In fact it follows from an earlier result of Henniart which is the most important consequence of his numerical local Langlands correspondence, and is used in the proof of the complete local Langlands correspondence. In practice we will be able to assume that each $\pi_{E_N,v}$ is of the form $\boxplus_i St(n_i,\chi_i)$ for a (unique) partition $n = \sum n_i$.

2. Automorphic representations of unitary groups.

Those are the ingredients that fit together to form the theory of automorphic representations. Now we turn to the CM field F. We consider cuspidal automorphic representations Π of $GL(n, F)$ – the symbol π is reserved for objects over F^+ – that satisfy the following three hypotheses:

(a) Π_v is cohomological for all archimedean v.
(b) $\Pi^\vee \xrightarrow{\sim} \Pi^c$ (globally, not just at archimedean places).
(c) For some finite prime v_0 of F^+ that splits as $w_0 w_0^c$ in F, Π_w is either supercuspidal or Steinberg for $w = w_0$ or $w = w_0^c$

I will explain vaguely how one can associate ℓ-adic Galois representations to such Π. The restrictions (a)-(c) are reflected in restrictive properties of the associated Galois representations. It should soon be possible to eliminate (c), thanks to the proof by Laumon and Ngo of the fundamental lemma for stable endoscopy for unitary groups, and work in progress in Paris. It remains a very convenient hypothesis for many reasons, but is inconvenient for certain applications. Various people have ideas for relaxing (b) – it's not necessary when F is imaginary quadratic (work of Taylor based on H-Taylor-Soudry). It's likely that (a) can be very slightly relaxed; this would be analogous to the Deligne-Serre results relating modular forms of weight one to two-dimensional complex representations with finite image (Artin representations). No one knows how to associate Artin representations of dimension > 2 to automorphic representations of any type, however; completely new methods are needed.

One should view the automorphic representation Π of $GL(n, F)$ as just one avatar of a more diffuse object. The L-function is another avatar, one that can also show up purely in terms of Galois theory. As indicated at the end of the previous section, hypothesis (b) implies that Π is a base change from some unitary group G over F. We construct the group G by choosing $G_v = G(F_v^+)$ for all places v of F^+. Let Σ^+ be the set of archimedean places of F^+, Σ a CM-type of F; i.e, for each $\sigma \in \Sigma^+$, an extension of σ to $\tau : F \to \mathbb{C}$. We fix $\sigma_0 \in \Sigma^+$, and let τ_0 be the corresponding element of Σ.

* We let $G_{\sigma_0} \simeq U(1, n-1)$.
* For $\sigma \in \Sigma^+$, $\sigma \neq \sigma_0$, we let $G_\sigma = U(0, n)$.
* We let G_{v_0} be the multiplicative group of a division algebra over $F_{v_0}^+$ with Hasse invariant $\frac{1}{n}$.
* If possible, we let G_v be quasi-split for all remaining (finite) v.

There may be a sign obstruction. In that case we have to replace F^+ by an appropriate totally real quadratic extension in which v_0 splits, and then impose the local condition at either or both primes above v_0, as necessary. We ignore this complication. On the other hand, there may be more than one G giving rise to the local G_v. This is called "failure of the Hasse principle" and one knows exactly when it takes place. We ignore this complication as well; any choice of G localizing to the given G_v will serve our purposes. Note that if $G(\mathbb{R}) = G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ has maximal compact subgroup K_∞, then $G(\mathbb{R})/K_\infty = U(1, n-1)/U(1) \times U(n-1)$ is isomorphic to the unit ball $B^{n-1} \subset \mathbb{C}^{n-1}$.

Automorphic representations of G are defined as before: they are irreducible $G(\mathbf{A})$-direct summands. The cohomological automorphic representations correspond to cohomology of compact arithmetic quotients of B^{n-1}, which are (usually smooth) projective varieties of dimension $n-1$. When G has been constructed, we find that

Theorem (Clozel-Labesse). *There exists an L-packet of cohomological automorphic representations π of G whose base change to $GL(n,F)$ equals* Π.

This has a precise meaning which one needs to understand in order to define base change as such. In particular, one needs to be able to define spherical representations of G_v for (almost all) v. But for the purposes of these lectures, it's enough to know the following facts:

(i) Suppose v splits as ww^c in F/F^+. Condition (b) guarantees that $\Pi_{w^c} = \Pi_w^\vee$ as representations of $GL(n,F_w) \xrightarrow{\sim} GL(n,F_{w^c})$. On the other hand because v is split we can identity $G_v = GL(n,F_w)$. Then for any π in the L-packet, $\pi_v = \Pi_w$ with respect to this identification.

(ii) If v is inert but Π_v is spherical then so is π_v and it can be determined by Satake transform for G_v.

(iii) If v is inert and Π_v is not spherical, or if v is ramified, then it's an open problem what π_v show up. In principle there is a finite set (local L-packet) and each element shows up. But this remains to be worked out.

(iv) If $v \in \Sigma^+$ but $v \neq \sigma_0$, then G_v is a compact unitary group and π_v is finite-dimensional. If $\Pi_v = \pi_{v,\tau}$ then $\pi_v = \tau^\vee$.

(v) Finally, if $v = \sigma_0$, the local L packet has n distinct elements. These have a geometric interpretation which is clearest if τ is the trivial representation, which we assume. We consider subgroups $\Gamma \subset G(F)$ that act discretely on the homogeneous space B^{n-1} for G_{σ_0}. For trivial τ, each π in the L-packet contributes to some cohomology space of the form $H^{n-1}(\Gamma\backslash B^{n-1},\mathbb{C})$, in a way I will attempt to explain momentarily. This has the Hodge decomposition

$$H^{n-1}(\Gamma\backslash B^{n-1},\mathbb{C}) = \oplus_{p+q=n-1} H^q(\Gamma\backslash B^{n-1},\Omega^p).$$

The n summands correspond precisely to the n distinct elements of the local L packet.

The appearance of the groups $H^{n-1}(\Gamma\backslash B^{n-1},\mathbb{C})$ is a clue to our interest in automorphic forms on unitary groups. The quotient $\Gamma\backslash B^{n-1}$ is actually one connected component of a more fundamental object, the *Shimura variety* associated to G and the choice of a compact open subgroup $K \subset G(\mathbf{A}_f)$:

$$Sh_K(G) = G(F)\backslash B^{n-1} \times G(\mathbf{A}_f)/K;\ Sh(G) = \varprojlim_K Sh_K(G)$$

This is a bit of a cheat (one needs to work with unitary similitude groups rather than unitary groups, and really one should let K vary) but I let that slide. For each K, Shimura variety $Sh_K(G)$ is naturally identified with a finite union of quotients $\Gamma\backslash B^{n-1}$, each of which is a smooth complex projective algebraic variety (for K sufficiently small); moreover, $Sh(G)$ can be viewed (canonical models) as a variety over a number field, which in practice will be F itself. Now

$$H^{n-1}(Sh(G),\mathbb{Q}_\ell) = \varinjlim_K H^{n-1}(Sh_K(G),\mathbb{Q}_\ell)$$

inherits a natural right action of the group $G(\mathbf{A}_f)$, but its primary advantage over complex cohomology is that it also carries an action of $Gal(\overline{\mathbb{Q}}/F)$ that commutes with $G(\mathbf{A}_f)$. Fixing π as before, and letting $* = \mathbb{C}$ or $* = \mathbb{Q}_\ell$, we define

$$H^{n-1}(Sh(G),*)[\pi] = Hom_{G(\mathbf{A}_f)}(\pi_f, H^{n-1}(Sh(G),*)).$$

This space is then n-dimensional (or is expected to be; we can treat it as if it were). When $* = \mathbb{Q}_\ell$, it carries the n-dimensional ℓ-adic Galois representation $\rho_{\ell,\pi}$ that is "associated" to our original Π, as I will explain in the next talk. When $* = \mathbb{C}$ it has the Hodge decomposition I mentioned before as a sum of n pieces.

The Galois representations $\rho_{\ell,\pi}$ are the third avatar of the automorphic representation Π of $GL(n,F)$. I conclude today with the fourth and final avatar. This is an automorphic representation of yet another unitary group G'. It is determined by setting $G'_v = G_v$ for all places except $v = \sigma_0$ and possibly $v = v_0$ (which you recall should really be considered a choice of two places if necessary). At σ_0, as at all other archimedean σ, we want $G'_{\sigma_0} = U(0,n)$. At v_0 we let G'_{v_0} be whatever it needs to be. Now $G'(\mathbb{R})$ is compact. The theorem of Clozel and Labesse applies to G' as well as to G. The resulting L-packet on G', labelled π', has the same properties as $\{\pi\}$ on G, except that there is no condition (v): condition (iv) now applies to all archimedean places.

The group G is needed to construct Galois representations, and is the main object of my book with Taylor. The group G' is needed to carry out the analysis of deformations of these Galois representations, as in the work of Taylor-Wiles. This will be discussed next time.

II. Properties of automorphic Galois representations

3. The Fontaine-Mazur conjecture.

I will begin by placing the results I intend to discuss in their standard conjectural framework. Conjectures associating automorphic representations to Galois representations work in both directions. The most convenient version for our purposes associates an individual ℓ-adic representation to an individual automorphic representation, and vice versa, and is due to Fontaine and Mazur. As in the previous lecture, F is a CM field, F^+ its maximal totally real subfield, E is a general number field.

In what follows an ℓ-adic representation is a representation on a finite dimensional vector space over $\overline{\mathbb{Q}}_\ell$, in practice over a fixed finite extension of $\mathbb{Q}_\ell$. Let S_ℓ denote the set of primes of E dividing ℓ. Following Fontaine and Mazur, we say an n-dimensional ℓ-adic representation ρ of $G_E = Gal(\bar{E}/E)$ is *of geometric type* if it is unramified outside the finite set $S \coprod S_\ell$ of primes of E, where $S \cap S_\ell = \emptyset$; and if at every $v \in S_\ell$ it has Fontaine's de Rham property. I will return to this property; for the moment it suffices to mention that it allows us to associate a set of Hodge-Tate numbers $h^p(\rho)$ to ρ, with $n = \sum h^p(\rho)$ (varying with v in general).

Following Clozel, we define an automorphic representation π of $GL(n, F)$ if π_v has integral infinitesimal character for every archimedean v. This means it can be associated to a Hodge structure.

Conjecture. *(a) Let ρ be an irreducible n-dimensional ℓ-adic representation of G_E of geometric type. Then there is a cuspidal automorphic representation of $GL(n, E)$ π_ρ of algebraic type associated to ρ, in the sense that $L(s, \pi_\rho) = L(s, \rho)$ where the former is the L-function associated by automorphic theory. In particular, $L(s, \rho)$ has an analytic continuation to an entire function satisfying the usual sort of functional equation.*

(b) Conversely, if π is an automorphic representation of $GL(n, E)$ of algebraic type, then there exists an ℓ-adic representation ρ_π of geometric type associated to π.

I need to explain $L(s, \rho)$, and the de Rham property. I also need to explain what happens as ℓ varies. So I will concentrate for the moment on Galois representations. First of all, ρ is assumed to be unramified outside $S \coprod S_\ell$. Let v be a prime of E outside the ramification set, and let $\Gamma_v \subset G_E$ be a decomposition group for v: $\Gamma_v \xrightarrow{\sim} Gal(\bar{E}_v/E_v)$ for some algebraic closure $\bar{E}_v$ of E_v, and let $I_v \subset \Gamma_v$ be the inertia group. Thus

$$\Gamma_v/I_v \xrightarrow{\sim} Gal(\bar{k}_v/k_v)$$

where k_v is the residue field at v. To say that ρ is unramified at v is to say that $\rho_v = \rho\,|_{\Gamma_v}$ factors through $\Gamma_v/I_v = Gal(\bar{k}_v/k_v)$. Inside this group is the arithmetic Frobenius ϕ_v that takes $a \in \bar{k}_v$ to a^{Nv}, where $Nv = |k(v)|$. Let $Frob_v = \phi_v^{-1}$ be geometric Frobenius, and define

$$L_v^{mot}(s, \rho) = \det(I - \rho(Frob_v)Nv^{-s})^{-1}.$$

This is the reciprocal of a polynomial of degree $n = \dim \rho$ in Nv^{-s}. The superscript mot indicates that this is not the right normalization for comparison with $L(s, \pi)$. We define

$$L^{mot,S}(s, \rho) = \prod_{v \notin S \coprod S_\ell} L_v^{mot}(s, \rho).$$

Meanwhile, the expression for $L_v^{mot}(s, \rho)$ contains at least one inconsistency: $\rho(Frob_v)$ is an $n \times n$ matrix over $\overline{\mathbb{Q}}_\ell$, whereas Nv^{-s} is a complex valued function of s. The rules allow us to choose an isomorphism $\overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, which would at least formally give us a complex infinite product, but how would we know which isomorphism to choose? The Fontaine-Mazur conjecture comes to the rescue: once ρ is of geometric type, it is conjectured automatically that there is a number field L such that ρ is in fact defined over L, in the sense that the characteristic polynomial of $\rho(Frob_v)$ has coefficients in L for all $v \notin S \coprod S_\ell$. So in fact we only need to choose an embedding $\iota : L \hookrightarrow \mathbb{C}$. There are finitely many such embeddings; we assume one has been chosen. The resulting L-function depends on the choice, and in fact for each such choice there is supposed to be a different π, but I'll say no more about that.

Fontaine-Mazur also conjectures that if ρ is irreducible, which we assume, there is an integer w, called the weight of ρ, such that, for any unramified v, the eigenvalues $\alpha_{i,v}$ of $\rho(Frob_v)$, which lie in a finite extension of L, have the property that, for any embedding $\iota : \bar{L} \hookrightarrow \mathbb{C}$,

$$|\iota(\alpha_{i,v})| = (Nv)^{\frac{w}{2}}.$$

Don't worry about L and w: they will always exist for the ρ and π we consider! We then define

$$L^S(s, \rho) = \det(I - \rho(Frob_v)(Nv)^{-\frac{w}{2}} Nv^{-s})^{-1} = L^{mot,S}(s + \frac{w}{2}, \rho).$$

The matrices $\rho(Frob_v)$, which are in fact defined only up to conjugation, cannot be chosen arbitrarily. Part of the Fontaine-Mazur conjecture is that the set of equivalence classes of ρ of geometric type is countable, and anyway, they are supposed to correspond to automorphic representations (of algebraic type). The Chebotarev density theorem, together with the Brauer-Nesbitt theorem, implies that the traces of $\rho(Frob_v)$ determine the composition series of ρ, and in particular determine what happens at $v \in S \coprod S_\ell$. There is a simple recipe for the local factor $L_v(s, \rho)$ for $v \in S$, completely analogous to the definition of the missing factors for the Artin L-function. There is a much more complicated recipe, due to Fontaine, for $v \in S_\ell$. This is where the de Rham property comes in, and I'll say something about this later. There is even a recipe, due to Serre, for the Gamma factors that should complete the L-function and satisfy a functional equation. This brings us back to the representation Π of the last lecture. Recall that F is a CM field. We denote $\mathbb{Q}_\ell(1) = \mathbb{Q}_\ell \otimes \varprojlim_m \mu_{\ell^m}$ the $G_v = Gal(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$-module with the cyclotomic character, $\mathbb{Q}_\ell(r) = \mathbb{Q}_\ell(1)^{\otimes r}$; for r negative we take the $-r$ tensor power of the dual.

Theorem (Kottwitz, Clozel, H-Taylor). *Let Π be a cuspidal automorphic representation of $GL(n, F)$ that satisfies*

(a) *Π_v is cohomological for all archimedean v (in particular, Π is of algebraic type).*

(b) $\Pi^\vee \xrightarrow{\sim} \Pi^c$.

(c) *For some finite prime* v_0 *of* F^+ *that splits as* $w_0 w_0^c$ *in* F, Π_w *is either supercuspidal or Steinberg for* $w = w_0$ *or* $w = w_0^c$.

Then

(i) *There is a number field* L *and, for any prime* ℓ *and any prime* λ *of* L *dividing* ℓ, *an* n*-dimensional representation* $\rho = \rho_{\Pi,\lambda} : G_F \to GL(n, L_\lambda)$, *unramified outside* $S \coprod S_\ell$, *where* S *is the set of places of* F *at which* Π *is ramified, such that*

$$L(s, \Pi) = L(s, \rho)$$

as Euler products over places of F.

(ii) *For any* $v \notin S_\ell$, *the semisimplification of* $\rho \mid \Gamma_v$ *is the Galois representation associated to* Π_v *by the local Langlands correspondence (this includes the assertion about the identity of Euler products, and also accounts for condition (c)).*

(iii) *One can replace "semisimplification" by "Frobenius semisimplification" (recent improvement by Taylor-Yoshida).*

(iv) *At* $v \in S_\ell$, ρ *is de Rham; if* Π *is unramified at* v, ρ *is even crystalline. The Hodge-Tate weights of* ρ *are determined by* Π_∞. *In the simplest case where* Π *contributes to cohomology with trivial coefficients, the Hodge-Tate weights of* ρ *at any* v *in* S_ℓ *are* $0, 1, \ dots, n-1$, *each with multiplicity one. In any case, the Hodge-Tate weights have multiplicity one.*

(v) *Condition (b) is reflected in the fact that* ρ *is* **polarized**: *there is a nondegenerate* G_F*-equivariant pairing*

$$\rho \otimes \rho^c \ \ \rightarrow \ \ \mathbb{Q}_\ell(1-n).$$

with respect to which the coefficient field is hermitian. Here ρ^c *is just* $\rho \circ c$.

The field L *can be taken to be a field of definition of any of the representations* π *of the unitary group* G *introduced the last time. The representation* $H^{n-1}(Sh(G), L_\lambda)[\pi] = Hom_{G(\mathbf{A}_f)}(\pi_f, H^{n-1}(Sh(G), L_\lambda))$ *defined the last time is (up to semisimplification) isomorphic to a finite number of copies of the representation* $\rho_{\Pi,\lambda}$. *If* Π *has cohomology with trivial coefficients, then* $w = n-1$.

The last paragraph explains why ρ is de Rham: any representation in ℓ-adic cohomology is de Rham (theorem first proved in most cases by Fontaine-Messing, then generalized by Faltings, completed by Tsuji). The proof ot this theorem follows the plan of Langlands' conjectures on Shimura varieties. The theorem for the Shimura variety, with $L_v(s, \rho)$ determined for almost all v, is due to Kottwitz. The connection with $GL(n)$ is due to Clozel. The complete determination of $L_v(s, \rho)$ and the result in (ii) is the main theorem of my book with Taylor.

In the next section I will explain the de Rham condition in more detail.

4. Elements of p-adic Hodge theory.

For reasons having to do with the origin of our work, ℓ plays the role of p. Crystalline ℓ-adic representations, or more generally de Rham ℓ-adic representations, are the kinds of representations that arise in the ℓ-adic etale cohomology of algebraic varieties over number fields or ℓ-adic fields. This is a theorem, however: the condition of being crystalline or de Rham has an abstract definition, due to Fontaine, whose most important properties have been established in just the last few years. The Shimura varieties of interest to us are smooth projective varieties and I will only consider places $v \in S_\ell$ where they have good reduction, so the representations will be in fact crystalline.

To make everything simpler, we assume $F_v = \mathbb{Q}_\ell$, although this is certainly not required. We fix a level K and also assume $Sh_K(G,*)$ has good reduction at v, i.e. there is a proper smooth scheme $\mathbb{S}_K(G,*)$ over $Spec(\mathcal{O}_v)$. Then the Shimura variety $Sh_K(G,*)$ gives rise to two spaces of cohomology over $\mathbb{Q}_\ell$. The first is the ℓ-adic cohomology $H_\ell = H^{n-1}(Sh_K(G,*),\mathbb{Q}_\ell) = \varprojlim_N H^{n-1}_{et}(Sh_K(G,*),\mathbb{Z}/\ell^N/\mathbb{Z}) \otimes \mathbb{Q}_\ell$. The second is de Rham cohomology:

$$H_{dR} = H^{n-1}_{dR}(\mathbb{S}_K/Spec(\mathcal{O}_v)) = \mathbb{H}^{n-1}(\mathbb{S}_K, \Omega^\bullet_{\mathbb{S}_K/\mathcal{O}_v}).$$

The former is topological, the latter is computed by differential forms with coefficients in $\mathcal{O}_v$. Both have the same finite dimension and the eigenspaces for Hecke operators on the two spaces have the same dimension as well. However, they have different structures. The ℓ-adic cohomology carries an action of G_F, and in particular of the decomposition group G_{F_v}. The second has two structures: a crystalline Frobenius:

$$\phi : H_{dR} \to H_{dR}$$

which is a bijective map that is $Frob$-linear

$$\phi(av) = Frob_v(a)\phi(v).$$

This doesn't look like anything more than a linear map but in fact it has the same property after base change to the completion of the maximal unramified extension of F_v. And a Hodge filtration: there is a filtration $\ldots F^pH_{dR} \subset F^{p-1}H_{dR} \ldots$ with

$$F^p/F^{p+1} \xrightarrow{\sim} H^q(\mathbb{S}_K, \Omega^{p-1}).$$

These two structures interact ("the Newton polygon lies above the Hodge polygon") but we don't need to know that.

What we do need to know, at least for a few seconds, is that there is a way to obtain H_{dR} from H_ℓ, with all the structure, and vice versa. The following theorem contains a part of ℓ-adic Hodge theory, and is due to many people.

Theorem. *There are fields $B_{crys} \subset B_{dR}$ containing $\mathbb{Q}_\ell^{unr}$, the maximal unramified extension of $\mathbb{Q}_\ell$, with compatible actions of $G_v = Gal(\overline{\mathbb{Q}}_\ell/F_v)$, and with the following additional structures:*

(1) *B_{dR} is a complete discrete valuation field, containing $\overline{\mathbb{Q}}_\ell$ and with residue field $\mathbb{C}_\ell$, the completion of $\overline{\mathbb{Q}}_\ell$ (via the residue map), and*
(a) *The valuation defines a G_v-stable (decreasing) filtration Fil^iB_{dR};*
(b) *There is a map $\mathbb{Q}_\ell(1) \to Fil^1B_{dR}$ of G_v-modules whose image contains a uniformizer;*

(c) $B_{dR}^{G_v} = F_v$
(2) B_{crys} *is a subring containing* $\mathbb{Q}_\ell(1)$ *and endowed with a* $Frob$*-linear injective automorphism* ϕ *satisfying*
(a) ϕ *commutes with* G_v*;*
(b) $\phi(t) = \ell \cdot t$ *for* $t \in \mathbb{Q}_\ell(1) \subset B_{crys} \cap Fil^1 B_{dR}$
(c) $Fil^0 B_{dR} \cap B_{crys}^{\phi=1} = \mathbb{Q}_\ell$
(d) $B_{crys}^{G_v} = F_v^0 := F_v \cap \mathbb{Q}_\ell^{unr}$

These fields, constructed by Fontaine according to an explicit and very complicated recipe, turn out to be of the highest importance for a huge variety of applications. They are called the ℓ-adic period rings (usually called p-adic, but not in our papers) because their main application is comparison between ℓ-adic topological cohomology and ℓ-adic de Rham (analytic) cohomology. We state the main theorem only for cohomology with trivial coefficients:

Theorem (Fontaine-Messing, Faltings, Tsuji). *(a) There is a natural isomorphism*

$$[H_\ell(Sh_K(G,*)) \otimes B_{dR}]^{G_v} \xrightarrow{\sim} H_{dR}(Sh_K(G,*)/F_v)$$

where G_v *acts diagonally and the filtration on the right-hand side is inherited from* B_{dR}*;*

(b) Suppose K *is sufficiently small that* $Sh_K(G,*)$ *is smooth. Assume* K *contains a hyperspecial maximal compact subgroup of* $G(F_w)$ *for all* w *dividing* ℓ*, e.g.* $G(F_w) = GL(n, F_w)$*, and* $K \supset GL(n, \mathcal{O}_w)$*. (This implies the existence of a smooth model over* $\mathcal{O}_v$*, as indicated above.) Then there is a natural isomorphism*

$$[H_\ell(Sh_K(G,*)) \otimes B_{crys}]^{G_v} \xrightarrow{\sim} H_{dR}(Sh_K(G,*)/F_v)$$

(actually with crystalline cohomology) and the action of ϕ *on the right-hand side is inherited from* B_{crys}*.*

(c) Assume $\ell > n$ *and* ℓ *is unramified in* F*. Then (b) is even true integrally: there is a* $\mathbb{Z}_\ell^{unr}$*-subalgebra* $A_{crys} \subset B_{crys}$ *and the isomorphism in (b) is valid over* $\mathcal{O}_v$ *(in a modified sense, see below)*

In (c), it is not true that

$$[H^{n-1}(Sh_K(G,*), \mathbb{Z}_\ell) \otimes A_{crys}]^{G_v} \xrightarrow{\sim} H_{dR}(Sh_K(G,*)/\mathcal{O}_v).$$

because $H^{n-1}(Sh_K(G,*), \mathbb{Z}_\ell)$ is not the right lattice. What we need to know is that one can construct a lattice $M_{crys}(Sh_K(G,*)) \subset H_{dR}(Sh_K(G,*)/F_v)$ as a union of the G_v-invariants in certain lattices in $H^{n-1}(Sh_K(G,*), \mathbb{Q}_\ell) \otimes A_{crys}$, and that the reduction modulo ℓ^m of M_{crys} for all m is a *Fontaine-Laffaille module.* This means that M_{crys} is a $\mathcal{O}_v$-module of finite type with a decreasing filtration $Fil^i M_{crys}$, with $Fil^0 M = M$, $Fil^\ell M = 0$, and a family of $Frob$-linear maps

$$\phi^i : Fil^i(M_{crys}) \to M_{crys}$$

such that for all i

$$\phi^i \mid_{Fil^{i+1}} = \ell\phi^{i+1},$$

and $M = \sum_i Im(\phi^i)$. The ϕ^i are derived from $\ell^{-i} \cdot \phi$ and the lattice is the smallest one for which this makes sense. We will only use the Fontaine-Laffaille property to define the deformation ring and to determine its numerical invariants.

5. The universal deformation ring.

In case I've forgotten to mention this, ℓ is henceforth an odd prime. We begin with the n-dimensional ℓ-adic representation ρ. For the remainder of these lectures I will usually assume it has coefficients in $\mathbb{Q}_\ell$, to simplify the exposition; this is by no means a necessary hypothesis. Since the Galois group is compact, ρ stabilizes a lattice, say $\Lambda \subset \mathbb{Q}_\ell^n$. Let $\bar{\rho}$ denote the representation on $\Lambda/\ell\Lambda$. This is an n-dimensional representation of G_F with coefficients in $\mathbb{F}_\ell$. A priori it depends on the choice of Λ, but we will always assume

Hypothesis. *$\bar{\rho}$ is absolutely irreducible.*

Then the Brauer-Nesbitt theorem implies $\bar{\rho}$ is independent of the choice of lattice, up to equivalence. The residual representation $\bar{\rho}$ is the basic object that allows us to define the universal deformation ring $R_{\bar{r}}$. One could work with the n-dimensional representation $\bar{\rho}$ itself, but the additional structure coming from the polarization is essential. We let $\mathcal{G}_n$ denote the algebraic group (group scheme over $\mathbb{Z}$) whose identity component $\mathcal{G}_n^o$ is $GL(n) \times GL(1)$, and which is a semi-direct product of $GL(n) \times GL(1)$ by the group $\{1, j\}$ acting by

$$j(g, \mu)j^{-1} = (\mu^t g^{-1}, \mu), \; g \in GL(n), \mu \in GL(1).$$

There is a homomorphism $\nu : \mathcal{G}_n \to GL(1)$ sending (g, μ) to μ and j to -1. We let $\mathfrak{g}_n = Lie(GL(n)) \subset Lie(\mathcal{G}_n)$, $\mathfrak{g}_n^0$ the trace zero subspace.

We consider a topological group Γ with a closed subgroup Δ of index 2 and an element $c \in \Gamma - \Delta$ with $c^2 = 1$.

Lemma. *Let R be any commutative ring. There is a natural bijection between the following two sets:*

1. Homomorphisms $r : \Gamma \to \mathcal{G}_n(R)$ such that $\Delta = r^{-1}\mathcal{G}_n^o(R)$,

2. Pairs $(\rho, <,>)$, where $\rho : \Delta \to GL(n, R)$ and

$$<,>: R^n \otimes R^n \to R$$

is a perfect bilinear pairing such that

- *$< x, y >= -\mu(c) < y, x >$ for some $\mu(c) \in R$, for all $x, y \in R^n$, and*
- *$\mu(\delta) < \delta^{-1}x, y >=< x, c\delta c y >$ for any $\delta \in \Delta$, some $\mu(\delta) \in R^\times$. Under this correspondence $\mu(\gamma) = \nu \circ r(\gamma)$ for all $\gamma \in \Gamma$.*

We let ω denote the cyclotomic character acting on $\mathbb{Q}_\ell(1)$ or $\mathbb{F}_\ell(1)$.

Corollary. *Let $k = \mathbb{Q}_\ell$ (resp. $\mathbb{F}_\ell$). There is a homomorphism*

$$r : G_{F^+} \to \mathcal{G}_n(\mathbb{Q}_\ell)$$

(resp. $\bar{r} :: G_{F^+} \to \mathcal{G}_n(\mathbb{F}_\ell)$ such that $r \mid_{G_F} = \rho$ (resp. $\bar{\rho}$), $\nu \circ r \mid_{G_F} = \omega^{1-n}$, $r(c) \in \mathcal{G}_n(k) - GL(n,k)$.

The possible extensions r of ρ are classified up to isomorphism by elements of $k^\times/(k^\times)^2$. We will ignore this issue.

Now $k = \mathbb{F}_\ell$ (though this may not always be legitimate), $\mathcal{O} = \mathbb{Z}_\ell$. Let S and S_ℓ be as before. We write $\Gamma = G_{F^+,S}$, $\Delta = G_{F,S}$, where the subscript S means "the Galois group of the maximal extension unramified outside $S \coprod S_\ell$. Let c denote complex conjugation, and assume $\bar{r}$ is absolutely irreducible. For $v \in S$ we let Δ_v be the decomposition group G_v. All places in S and S_ℓ are assumed split in F/F^+. We write $S = S_1 \cup S_2$, where at places in S_1 (resp. S_2) Π_v is supercuspidal (resp. Steinberg). Later we will add an extra set Q and where Π_v is unramified for $v \in Q$. For $v \in S_1$, $\rho_v = \rho \mid_{\Delta_v}$ is then absolutely irreducible. We assume for the moment that the residual representation $\bar{\rho}_v$ is also absolutely irreducible. Conditions at $v \in S_2 and Q$ will be specified in the next lecture. Let $\mathcal{C}^f_\mathcal{O}$ be the category of Artinian local $\mathcal{O}$-algebras A for which the map $\mathcal{O} \to A$ induces an isomorphism on residue fields, $\mathcal{C}_\mathcal{O}$ the full subcategory of topological $\mathcal{O}$-algebras whose objects are inverse limits in $\mathcal{C}^f_\mathcal{O}$. For A an object of $\mathcal{C}^f_\mathcal{O}$ or $\mathcal{C}_\mathcal{O}$ we want to classify liftings of $\bar{\rho}$ to homomorphisms $\rho' : \Delta \to GL(n,A)$ satisfying the properties of 2 of the Lemma, or more properly homomorphisms $r' : \Gamma \to \mathcal{G}_n(A)$ lifting $\bar{r}$. Moreover, we only consider liftings up to equivalence: two liftings are *equivalent* if they are conjugate by an element of $GL(n,A)$ that reduces to 1 in $GL(n,A/m_A) = GL(n,k)$, where m_A is the maximal ideal of A.

Suppose A is an object of $\mathcal{C}_\mathcal{O}$ with closed ideal I, and suppose r_1 and r_2 are two liftings of $\bar{r}$ to A that are equivalent mod I. By induction on the length of A/I we can reduce to the case where $m_A \cdot I = (0)$. Thus there is a short exact sequence

$$1 \to M(n,k) \to \mathcal{G}_n(A) \to \mathcal{G}_n(A/I) \to 1$$

where $M(n,k) = 1 + M(n,I) \subset \mathcal{G}_n(A)$. Then

$$\gamma \mapsto r_2(\gamma) r_1(\gamma)^{-1} - 1 \in M(n,k)$$

defines a cocycle $[r_2 - r_1] \in Z^1(\Gamma, M(n,k))$ where the action of Γ on $M(n,k)$ is given by conjugation in $\mathcal{G}_n(A)$, i.e. by $ad\ \bar{r}$. We have a cocycle because the liftings are group homomorphisms; and two cocycles give rise to equivalent liftings if and only if they define the same class in $H^1(\Gamma, ad\ \bar{r})$.

Without much difficulty we can prove that the functor classifying liftings of $\bar{r}$ to $\mathcal{C}_\mathcal{O}$ is representable by a ring R^{univ}, in the sense that homomorphisms $R^{univ} \to A$ are canonically in bijection with liftings of $\bar{r}$ to A. Since $\Gamma = G_{F^+,S}$, the resulting liftings are automatically unramified outside $S \coprod S_\ell$. However, we need additional conditions, for example to guarantee that the liftings are geometric in the sense of Fontaine-Mazur. The only liftings of interest are thus those that satisfy certain conditions upon restriction to Δ_v, $v \in S \coprod S_\ell$. This makes representability more delicate. We begin with the minimal conditions. We always assume ρ comes from cohomology with trivial coefficients:

Hypotheses (minimal case). *We only consider liftings r' of $\bar{r}$ with the following properties:*

1. *For $v \in S_1$, the natural map $r'(I_v) \to \bar{r}(I_v)$ is an isomorphism.*
2. *For $v \in S_2$, wait until the next lecture*
3. *For $v \in S_\ell$, $r' \mid_{\Delta_v}$ is crystalline (Fontaine-Laffaille) with Hodge-Tate weights $0, 1, \ldots, n-1$, each with multiplicity one.*
4. *For the moment, Q is empty.*

Let $\rho' = r' \mid_\Delta$. We always assume

Polarization hypothesis. *We assume $\nu \circ \rho' = \omega^{1-n}$.*

Condition (3) means that that the Fontaine-Laffaille functor $M_{crys}(\rho')$ attached to $\rho' \mid_{\Delta_v}$ is a free A-module of rank n with Fil^i/Fil^{i+1} free of rank 1 for $i = 0, 1, \ldots, n-1$. One of the main open questions in the theory is what condition to use when $\ell < n$. If we restrict attention to ordinary representations, or even "nearly ordinary" in Hida's sense, there is a practical substitute. Otherwise, it's completely mysterious.

Theorem. *The functor classifying minimal liftings is representable in $\mathcal{C}_\mathcal{O}$ by a noetherian $\mathbb{Z}_\ell$-algebra $R^{min}_{\bar{r}}$ with residue field k.*

The proof, which follows the arguments of Mazur and Ramakrishna, is based on Schlessinger's criterion for pro-representability of functors on categories like $\mathcal{C}^f_\mathcal{O}$. In the next lecture I will say more about $R^{min}_{\bar{r}}$ and the non-minimal variants, and the relations with Galois cohomology and Selmer groups. The goal of the theory is to prove that $R^{min}_{\bar{r}}$ and its non-minimal variants are isomorphic to certain Hecke algebras, acting on automorphic forms on the *definite* unitary group G'. This is sufficient to prove that every lifting of $\bar{r}$ of geometric type, in the sense of Fontaine-Mazur, comes from automorphic forms on $GL(n, F)$.

III. Numerical invariants of deformation rings

6. Local Galois cohomology and Selmer groups.

I begin by introducing the conditions at primes in S_2. As before, Q is assumed empty. For each such v, we assume there is a positive integer m and a $\Delta_v = G_v$-invariant increasing filtration

$$0 = \overline{Fil}_v^m \subset \ldots \overline{Fil}_v^i \subset \overline{Fil}_v^{i-1} \subset \overline{Fil}_v^0 = \bar{\rho}.$$

We assume $\overline{gr}_v^i = \overline{Fil}_v^i / \overline{Fil}_v^{i+1}$ absolutely irreducible or zero for all i. Moreover, we assume there is a factorization $n = mr$ and an absolutely irreducible representation $\tilde{r}_v : G_v \to GL(r, \mathcal{O})$ and, for all i, isomorphisms

$$\alpha^i : \tilde{r}_v \otimes_{\mathcal{O}} k(i) \xrightarrow{\sim} \overline{gr}_v^i$$

of G_v-modules. For any j, we define

$$\overline{Fil}_v^j(ad(\bar{r})) = \{h \in Hom(\bar{r}, \bar{r}) \mid \forall i \ h(\overline{Fil}_v^i) \subset \overline{Fil}_v^{i+j}\}.$$

The liftings r of $\bar{r}$ to A are then assumed to have the same property: it is assumed that G_v preserves a filtration by free A-submodules $\ldots Fil_v^i \subset Fil_v^{i-1} \ldots$, mecessarily split as A-modules but not as G_v-modules. These can be classified cohomologically as before. We again consider an ideal I with $m_A \cdot I = 0$, and suppose there are two liftings $(r_1, Fil_{1,v}^*)$ and $(r_2, Fil_{2,v}^*)$ to A that agree modulo I. Suppose $B \in M(n, I)$ (i.e., in $1 + M(n, I)$) takes $Fil_{1,v}^*)$ to $Fil_{2,v}^*)$, and consider the cocycle on G_v:

$$\gamma \mapsto [r_2 - r_1](\gamma) + (1 - ad\ \bar{r}(\gamma)) \cdot B.$$

This is cohomologous to the restriction to G_v of $[r_2 - r_1]$ but takes values in $Fil_v^0(ad\ \bar{r}) \otimes_k I$ by construction. The class, denoted $[(r_2, Fil_{2,v}^*) - (r_1, Fil_{1,v}^*)]$, in $H^1(\Delta_v, Fil_v^0(ad\ \bar{r}) \otimes_k I)$ is independent of the choice of B and this cohomology group is in one-to-one correspondence with liftings to A that agree with $(r_1, Fil_{1,v}^*)$ modulo ℓ , up to conjugation by $1 + M(n, I)$.

With this example in mind, I am ready to present the general Galois cohomological formalism of deformation. We let $\mathcal{D}$ be a collection of data of the following form. For each $v \in S \coprod S_\ell$, let $L_v \subset H^1(\Delta_v, Fil_v^0(ad\ \bar{r}))$ be a k-subspace, and $\mathcal{D}_v$ a set of liftings of $(\bar{r}\mid_{\Delta_v}, \overline{Fil}_v^*)$ (i.e., just of $\bar{r}\mid_{\Delta_v}$ if $v \notin S_2$) to algebras in $\mathcal{C}_{\mathcal{O}}$ satisfying the following conditions:

1. $(k, \bar{r}\mid_{\Delta_v}, \overline{Fil}_v^*) \in \mathcal{D}_v$.
2. If $(A, r, Fil_v^*) \in \mathcal{D}_v$ and if $f : A \to A'$ is a morphism in $\mathcal{C}_{\mathcal{O}}$ then $(A', f \circ f, f(Fil_v^*)) \in \mathcal{D}_v$.
3. Suppose $(A_i, r_i, Fil_{i,v}^*) \in \mathcal{D}_v$ for $i = 1, 2$, $I_1 \in A_1$, $I_2 \in A_2$ closed ideals, with $f : A_1/I_1 \xrightarrow{\sim} A_2/I_2$ in $\mathcal{C}_{\mathcal{O}}$ that identifies the images of the chosen liftings mod I_i. Let

$$A_3 = \{(a_1, a_2) \in A_1 \times A_2 \mid f(a_1 \pmod{I_1}) = a_2 \pmod{I_2}\}.$$

Then the join $(A_3, r_1 \times r_2, Fil^*_{1,v} \times Fil^*_{2,v})$ is in $\mathcal{D}_v$.

(4) $\mathcal{D}_v$ is closed under projective limits.
(5) $\mathcal{D}_v$ is closed under equivalence.
(6) Suppose $A \in \mathcal{C}_{\mathcal{O}}$, $I \subset A$ a closed ideal with $m_A \cdot I = 0$. Suppose $(r_1, Fil^*_{1,v})$ and $(r_2, Fil^*_{2,v})$ are two liftings of $(\bar{r}, \overline{Fil}^*_v)$ to A with the same image modulo I, and suppose $(A, r_1, Fil^*_{1,v}) \in \mathcal{D}_v$. Then $(A, r_2, Fil^*_{2,v}) \in \mathcal{D}_v$ if and only if $[(r_2, Fil^*_{2,v}) - (r_1, Fil^*_{1,v})]$ lies in L_v.

We call $\mathcal{D}_v$ *liftable* if for any $I \subset A$ as above any lifting of $(\bar{r}, \overline{Fil}^*_v)$ to A/I admits a lifting to A. We call L_v *minimal* if

$$\dim_k L_v = \dim_k H^0(\Delta_v, Fil^0_v ad\ \bar{r}).$$

The condition that $\mathcal{D}_v$ be liftable is necessary for the universal deformation ring to be reasonable; the minimality condition is necessary for application of the Taylor-Wiles method. This will be explained here. The hypotheses we have imposed at S_1 and S_2 translate into definitions of spaces L_v as above. The functor $Def_{\mathcal{D}}$ is the functor that to A associates liftings (A, r, Fil^*_v) of type $\mathcal{D}$.

We define two new complexes. There are morphisms (localization maps)

$$C^\bullet(\Gamma, ad\ \bar{r}) \rightarrow \oplus_{v \in S_2} C^\bullet(\Delta_v, ad\ \bar{r}/Fil^0_v ad\ \bar{r}),$$

$$C^\bullet(\Gamma, ad\ \bar{r}) \rightarrow \oplus_{v \in S \coprod S_\ell} C^\bullet(\Delta_v, ad\ \bar{r})/L^\bullet_v$$

where $C^\bullet$ is the complex computing group cohomology (well defined up to quasi-isomorphism), L^1_v the preimage in $C^1(\delta_v, Fil^0 ad\ \bar{r})$ of the $L_v \in H^1$ introduced above, $L^0_v = C^0(\Delta_v, Fil^0_v ad\ \bar{r})$, and the other L^i_v are trivial. Let $C^\bullet(\Gamma, \overline{Fil}^*_v, ad\ \bar{r})$ denote the cone on the first localization map, $C^\bullet_{\mathcal{D}}(\Gamma, ad\ \bar{r})$ the cone on the second localization map. There are formally long exact sequences:

$$\ldots \rightarrow \oplus_v D^{i-1}_v \rightarrow H^i_{\mathcal{D}}(\Gamma, ad\ \bar{r}) \rightarrow H^i(\Gamma, ad\ \bar{r}) \rightarrow \oplus_v D^i_v \rightarrow H^{i+1}_{\mathcal{D}}(\Gamma, ad\ \bar{r}) \rightarrow \ldots$$

where D^i_v is the cohomology of the complex $C^i(\Delta_v, ad\ \bar{r})/L^i_v$, namely

$$D^0_v = \ker[H^0(\Delta_v, ad\ \bar{r}/Fil^0_v ad\ \bar{r}) \rightarrow H^1(\Delta_v, Fil^0_v ad\ \bar{r})/L_v],$$

$$D^1_v = H^1(\Delta_v, ad\ \bar{r})/p(L_v),\ D^2_v = H^2(\Delta_v, ad\ \bar{r}),\ D^i = 0, i > 2.$$

Here $p : H^1(\Delta_v, Fil^0 ad\ \bar{r}) \rightarrow H^1(\Delta_v, ad\ \bar{r})$ is the natural map. In practice, all the terms vanish for $i > 2$. Then

Lemma. *The Euler characteristic (alternating product of orders of groups) χ_v of $D^\bullet_v$ is*

$$\chi(\Delta_v, ad\ \bar{r}) \cdot |L_v| \cdot |H^0(\Delta_v, Fil^0_v ad\ \bar{r})|^{-1}$$

so

$$\chi_{\mathcal{D}}(\Gamma, ad\ \bar{r}) = \chi(\Gamma, ad\ \bar{r}) \cdot \prod_v \chi(\Delta_v, ad\ \bar{r})^{-1} \cdot |L_v|^{-1} \cdot |H^0(\Delta_v, Fil^0_v ad\ \bar{r})|.$$

Proof. The second statement follows immediately from the first. We drop the subscript v, and remember not to confuse the decomposition groups Δ_v with the global Galois group Δ. Consider the short exact sequence

$$0 \to H^0(\Delta, Fil^0) \to H^0(\Delta, ad\,\bar{r}) \to H^0(\Delta, ad\,\bar{r}/Fil^0) \xrightarrow{q} H^1(\Delta, Fil^0) \xrightarrow{p} H^1(\Delta, ad\,\bar{r}) \to ,$$

where $Fil^0 = Fil^0 ad\ \bar{r}$. Write $L^+ = p(L)$, $L^- = L \cap \ker(p) = L \cap im(q)$, so that $|L| = |L^+|\cdot|L^-|$. Write $h^i = |H^i(\Delta, ad\,\bar{r})|$. Dropping the terms h^2 that are common to the two sides, we need to verify that

$$|\ker[H^0(\Delta, ad\ \bar{r}/Fil^0) \to H^1(\Delta, Fil^0)/L]|/(h^1/|L^+|) = h^0|L|/h^1 \cdot |H^0(\Delta, Fil^0)|.$$

i.e.

$$|\ker[H^0(\Delta, ad\ \bar{r}/Fil^0) \to H^1(\Delta, Fil^0)/L]|/|L^-| = h^0/|H^0(\Delta, Fil^0)|).$$

But obviously

$$|\ker[H^0(\Delta, ad\,\bar{r}/Fil^0) \to H^1(\Delta, Fil^0)/L] = |\ker[H^0(\Delta, ad\,\bar{r}/Fil^0) \to H^1(\Delta, Fil^0)|\cdot|L^-|$$

so that we have to check

$$|\ker[H^0(\Delta, ad\ \bar{r}/Fil^0) \to H^1(\Delta, Fil^0)| = h^0/|H^0(\Delta, Fil^0)|$$

and this is obvious.

The importance of this calculation is contained in the following proposition:

Proposition. *The functor $Def_{\mathcal{D}}$ is pro-representable by an object $R_{\mathcal{D}}^{univ}$ of $\mathcal{C}_{\mathcal{O}}$. Let $\mathfrak{m}_{\mathcal{D}}$ be the maximal ideal of $R_{\mathcal{D}}^{univ}$. There is a canonical isomorphism*

$$Hom(\mathfrak{m}_{\mathcal{D}}/(\mathfrak{m}_{\mathcal{D}})^2, k) \xrightarrow{\sim} H^1_{\mathcal{D}}(\Gamma, ad\ \bar{r}).$$

Moreover, suppose each $\mathcal{D}_v$ is liftable and $H^2_{\mathcal{D}}(\Gamma, ad\ \bar{r}) = 0$ Then $R_{\mathcal{D}}^{univ}$ is a power series ring in $\dim_k H^1_{\mathcal{D}}(\Gamma, ad\ \bar{r})$ variables over $\mathcal{O}$.

Proof. I will skip the pro-representability, which consists in verifying Schlessinger's criterion; the difficulty (if any) comes from the filtrations. More important for the sake of the Taylor-Wiles method is the isomorphism. It is completely standard that $Hom(\mathfrak{m}_{\mathcal{D}}/(\mathfrak{m}_{\mathcal{D}})^2, k)$ is the Zariski tangent space to $R_{\mathcal{D}}^{univ}$, i.e. classifies $Def_S(k[\epsilon]/(\epsilon^2))$ (liftings to the dual numbers). If there were no local conditions, in particular no L_v, then we would just have the situation described last time: $A = k[\epsilon]/(\epsilon^2)$, $A/I = k$, and the isomorphism is just the identification of liftings (up to isomorphism) with cohomology classes in $H^1(\Gamma, ad\ \bar{r})$, as already discussed.

The proof with the conditions uses the definition of the cone complex

$$C^i_{\mathcal{D}}(\Gamma, ad\ \bar{r}) = C^i(\Gamma, ad\ \bar{r}) \oplus_v C^{i-1}(\Delta_v, ad\ \bar{r})/L^{i-1}$$

with boundary map

$$(\phi, (\psi_v)) \mapsto (\partial\phi, (\phi \mid_{\Delta_v} - \partial\psi_v)).$$

Now a lift $(r, (Fil_v^i))$ of $(\bar{r}, (\overline{Fil}_v^i))$ to $k[\epsilon]/(\epsilon^2)$ is of the form

$$r = \bar{r} + \phi\epsilon\bar{r},\ Fil_v^i = (1 + a_v\epsilon)\overline{Fil}_v^i + \epsilon\overline{Fil}_v^i)$$

where

$$\phi \in Z^1(\Gamma, ad\ \bar{r}),\ \ , a_v \in ad\ \bar{r}/Fil_v^0 ad\ \bar{r},$$

and the two satisfy

$$\phi = (ad\ \bar{r} - 1)a_v \in Z^1(\Delta_v, ad\ \bar{r}/Fil_v^0 ad\ \bar{r}).$$

The first condition is just what we have already seen and the condition on a_v just measures the difference with the trivial lifting $\overline{Fil}_v^i + \epsilon\overline{Fil}_v^i$, since the component in Fil_v^0 changes nothing. Finally, the relation between ϕ and a_v comes from the requirement that $r(\delta)$ fix the new filtration for $\delta \in \Delta_v$, i.e.

$$(1 + \phi(\delta)\epsilon)\bar{r}(\delta)(1 + a_v\epsilon\overline{Fil}_v^i) \subset (1 + a_v\epsilon)\overline{Fil}_v^i + \epsilon\overline{Fil}_v^i = (1 + a_v\epsilon)\bar{r}(\delta)\overline{Fil}_v^i + \epsilon\overline{Fil}_v^i$$

for all i and all $\delta \in \Delta_v$. Since $\epsilon^2 = 0$, only the coefficients of ϵ need to be matched, and this yields

$$[\phi\bar{r}(\delta) + \bar{r}(\delta)a_v - a_v\bar{r}(\delta)](\overline{Fil}^i) \subset \overline{Fil}^i\ \forall i, \delta$$

or in other words that the expression in brackets lies in $Fil_v^0 ad\ \bar{r}$, as does the same expression multiplied on the right by $\bar{r}\delta^{-1}$, and this gives the relation.

Now the combination of these relations shows that the collection $(\phi, (a_v))$ is a cocycle in $C^1_{\mathcal{D}}(\Gamma, ad\ \bar{r})$, ignoring the L_v for the moment. One checks that $(\phi, (a_v))$ and $(\phi', (a'_v))$ correspond to equivalent lifts if and only if there is $B \in ad\ \bar{r} = C^0(\Gamma, ad\ \bar{r}) = C^0_{\mathcal{D}}(\Gamma, ad\ \bar{r})$ such that $\phi' = \phi + (ad\ \bar{r} - 1)B$ and $a'_v = a_v + B$. Thus the cohomology group is in bijection with liftings. Finally, property (6) of $\mathcal{D}$ requires that the difference between the trivial lifting to $k[\epsilon]/(\epsilon^2)$ and any given lifting, which is what is measured by $(\phi, (a_v))$, has to lie in L_v locally at all $v \in S \coprod S_\ell$, i.e. that

$$\phi - (ad\ \bar{r} - 1)a_v \in L_v$$

and not only in Fil_v^0. This is precisely what is measured by the cone complex.

The final assertion is standard in deformation theory. Let A and I be as before, $(r, Fil^\bullet) \in Def_{\mathcal{D}}(A/I)$. Assuming each $\mathcal{D}_v$ is liftable, one chooses lifts r_v of $r \mid_{\Delta_v}$ from A/I to I for each v (along with the filtrations, where appropriate); one also chooses a set theoretic lift $\tilde{r} : \Gamma \to \mathcal{G}_n(A)$ of r. The obstruction to the set-theoretic lift being a homomorphism defines a class in $H^2(\Gamma, ad\ \bar{r})$ by a standard procedure, and the obstruction to it being a homomorphism whose restriction to each Δ_v is conjugate to r_v belongs to $H^2_{\mathcal{D}}(\Gamma, ad\ \bar{r})$. So under the hypotheses, there is never a restriction to lifting, and this implies that $R^{univ}_{\mathcal{D}}$ is formally smooth, which translates to the final claim.

Now we want to use the Euler characteristic lemma to calculate this dimension. We need some results from duality:

Theorem (local duality). *Let v be a non-archimedean place of F^+. Cup-product defines a non-degenerate pairing*

$$H^i(\Delta_v, ad\ \bar{r}(1)) \otimes H^{2-i}(\Delta_v, ad\ \bar{r}) \to H^2(\Delta_v, \mu_\ell) \xrightarrow{\sim} \mathbb{F}_\ell.$$

This is a version of class field theory and is true for more general representations of Δ_v. See Milne's *Arithmetic Duality Theorems*, I, Corollary 2.3. In particular, it shows that $H^i(\Delta_v, *) = 0$ for $i > 2$, and it allows us to define $L_v^\perp$ as the annihilator in $H^1(\Delta_v, ad\ \bar{r}(1))$ of L_v with respect to the above pairing.

As long as $\ell > 2$, the groups $H^i_{\mathcal{D}}(\Gamma, *) = 0$ for $i > 3$ and $H^i(\Gamma, *) = 0$ for $i > 2$ (a consequence of global class field theory). The Euler characteristic lemma thus applies to our situation. We need formulas for $\chi(\Gamma, ad\ \bar{r})$ (which can be defined as $|H^0||H^2|/H^1$ even when $\ell = 2$) and for $\chi(\Delta_v, ad\ \bar{r})$. The following theorems are due to Tate and are stated as Theorems I.2.8 and I.5.1 in Milne's book:

Theorem. *(a) Local Euler characteristic: Suppose $v \in S \coprod S_\ell$ is of residue characterstic p. Then*

$$\chi(\Delta_v, ad\ \bar{r}) = |ad\ \bar{r}|^{[F_v^+:\mathbb{Q}_p]}.$$

(b) Global Euler characteristic

$$\chi(\Gamma, ad\ \bar{r}) = \prod_{v|\infty} |H^0(\Delta_v, ad\ \bar{r})|/|ad\ \bar{r}|_v.$$

or equivalently

$$h^0 - h^1 + h^2 = \sum_{v|\infty} (\dim_k (ad\ \bar{r})^{c_v=1} - n^2).$$

Since all v are real in (b), $|ad\ \bar{r}|_v = |ad\ \bar{r}|$ (at complex places it would be the square). We use this calculation to determine $\chi_{\mathcal{D}}(\Gamma, ad\ \bar{r})$, which we write logarithmically (base ℓ), with a minus sign, as $-h^0_{\mathcal{D}} + h^1_{\mathcal{D}} - h^2_{\mathcal{D}} + h^3_{\mathcal{D}}$. It is obvious that

$$h^0_{\mathcal{D}} = h^0(\Gamma, ad\ \bar{r}) = \dim(h^0(\Delta, ad\ \bar{r})^{c=1}) = \dim(center)^{c=1} = 0$$

the next to last equality being Schur's lemma, and the last following because c acts by $X \mapsto -{}^tX$. Combining these calculations with the Lemma above, we find:

$$\begin{aligned} h^1_{\mathcal{D}} - h^2_{\mathcal{D}} + h^3_{\mathcal{D}} = & \\ & \sum_{v|\infty} (n^2 - \dim_k (ad\ \bar{r})^{c_v=1}) + \sum_{v \in S} \dim_k L_v - h^0(\Delta_v, ad\ \bar{r}) \\ & + \sum_{v \in S_\ell} \dim_k L_v - h^0(\Delta_v, ad\ \bar{r}) - n^2[F_v : \mathbb{Q}_\ell] \end{aligned}$$

where we are writing $h^i(*)$ for $\dim_k H^i(*)$ where possible. Cancelling the $\sum_{v|\infty} n^2$ against the $\sum_{v|\ell} n^2[F_v : \mathbb{Q}_\ell]$ this simplifies to

Corollary.

$$h^1_{\mathcal{D}} - h^2_{\mathcal{D}} + h^3_{\mathcal{D}} = \sum_{v \in S \coprod S_\ell} [\dim_k L_v - h^0(\Delta_v, ad\ \bar{r})] - \sum_{v \mid \infty} (\dim_k (ad\ \bar{r})^{c_v=1})$$

More duality and local calculations are needed to simplify this expression. We define

$$H^1_{\mathcal{L}^\perp}(\Gamma, ad\ \bar{r}(1)) = \ker[H^1(\Gamma, ad\ \bar{r}(1)) \to \oplus_{v \in S \coprod S_\ell} H^1(\Delta_v, ad\ \bar{r}(1))/L_v^\perp.$$

Lemma.

(1) $h^2_{\mathcal{D}} = \dim_k H^1_{\mathcal{L}^\perp}(\Gamma, ad\ \bar{r}(1))$.
(2) $h^3_{\mathcal{D}} = \dim_k H^0(\Gamma, ad\ \bar{r}(1))$.
(3) *For all* $v \in S_\ell$,

$$\dim_k L_v - \dim_k H^0(\Delta_v, ad\ \bar{r}) = n(n-1)[F_v : \mathbb{Q}_\ell]/2.$$

(4) *For all* $v \mid \infty$, *there is a constant* $c_v = \pm 1$ *such that, ,*

$$\dim_k (ad\ \bar{r})^{c_v=1} = n(n-1)/2 + n(1+c_v)/2.$$

The first two equalities are consequences of duality and the Poitou-Tate exact sequence. The third is derived from the Fontaine-Laffaille theory. The fourth is a simple computation using the structure of $\mathcal{G}_n$. We now introduce the mysterious set Q, also of primes split in F/F^+, disjoint from $S = S_1 \cup S_2$. We choose conditions $(\mathcal{D}_v, L_v)$ for $v \in Q$, and let $\mathcal{D}(Q)$ be the new set of deformation conditions; $h^1_{\mathcal{L}(Q)^\perp}$ is defined by including these conditions. Combining these calculations with the previous corollary, and again cancelling the terms common to the primes above ℓ and ∞, and introducing the mysterious set Q, we find

Corollary. $\dim_k \mathfrak{m}_{\mathcal{D}(Q)}/(\mathfrak{m}_{\mathcal{D}(Q)})^2 = h^1_{\mathcal{D}(Q)}$ *is given by*

$$h^1_{\mathcal{L}(Q)^\perp}(\Gamma, ad\ \bar{r}(1)) - h^0(\Gamma, ad\ \bar{r}(1)) - n \sum_{v \mid \infty} \frac{c_v + 1}{2} + \sum_{v \in S \cup Q} [\dim_k L_v - h^0(\Delta_v, ad\ \bar{r})].$$

In particular, suppose

(i) *All the* L_v *are minimal for* $v \in S_1 \cup S_2$
(ii) *For all* $v \in Q$, $\dim_k L_v - h^0(\Delta_v, ad\ \bar{r}) = 1$
(iii) *All the* $\mathcal{D}_v$ *are liftable*
(iv) $h^0(\Gamma, ad\ \bar{r}(1)) = h^2_{\mathcal{D}(Q)}(\Gamma, ad\ \bar{r}) = 0$

Then $R^{univ}_{\mathcal{D}(Q)}$ *is a power series over* $\mathcal{O}$ *in* $|Q| - n \sum_{v \mid \infty} \frac{c_v+1}{2}$ *variables.*

Proof. Recall that $h^2_{\mathcal{D}(Q)}(\Gamma, ad\ \bar{r}) = h^1_{\mathcal{L}(Q)^\perp}(\Gamma, ad\ \bar{r}(1))$. Thus the vanishing of $h^2_{\mathcal{D}(Q)}(\Gamma, ad\ \bar{r})$ means both that the deformation problem is unobstructed and that the main global term in the Euler characteristic is zero, leaving only the local terms.

At the end of the process, it will turn out that the relative dimension of $R^{univ}_{\mathcal{D}}$ is at least $|Q|$, which proves a posteriori that $c_v = -1$ for all real v. In the meantime, we simplify:

Lemma. *For all $v \in S_1 \cup S_2$, the L_v are minimal.*

Proof. This is a calculation in Galois cohomology. For $v \in S_2$ it's fairly complicated, so we will just consider S_1 to illustrate how this works. First we need to translate the hypothesis

$$r'_\ell(I_v) \xrightarrow{\sim} \bar{r}(I_v)$$

into a definition of L_v. There is a short inflation-restriction exact sequence

$$0 \to H^1(\Delta_v/I_v, ad\ \bar{r}^{I_v}) \to H^1(\Delta_v, ad\ \bar{r}) \to H^0(\Delta_v/I_v, H^1(I_v, ad\ \bar{r}))$$

and one verifies that our hypothesis translates into taking

$$L_v = Im(H^1(\Delta_v/I_v, ad\ \bar{r}^{I_v})).$$

The calculation is simplest if $\bar{r}$ remains absolutely irreducible upon restriction to I_v, and in particular to Δ_v. In that case,

$$h^0(\Delta_v, ad\ \bar{r}) = \dim_k Hom_{\Delta_v}(\bar{r}, \bar{r}) = 1$$

by Schur's Lemma. On the other hand, for the same reason,

$$ad\ \bar{r}^{I_v} = Hom_{I_v}(\bar{r}, \bar{r}) = Hom_{\Delta_v}(\bar{r}, \bar{r})$$

and so

$$L_v = H^1(\Delta_v/I_v, ad\ \bar{r}^{\Delta_v}) = Hom_k(\Delta_v/I_v, k)$$

has dimension 1. The general case is only slightly more difficult.

7. Prologue to Taylor-Wiles.

The set Q is the heart of the Taylor-Wiles method. We require in fact a set Q_N for each positive integer N. As above, $\mathcal{D}$ is the set of conditions $(\mathcal{D}_v, L_v)$ for $v \in S_1 \cup S_2$. Let

$$r = h^1_{\mathcal{L}^\perp}(\Gamma, ad\ \bar{r}(1)).$$

We want to choose sets Q_N, for all N, disjoint from S, such that

(1) $|Q_N| = r$
(2) $h^1_{\mathcal{L}(Q_N)^\perp}(\Gamma, ad\ \bar{r}(1)) = 0$
(3) For $v \in Q_N$, $|k(v)| \equiv 1 \pmod{\ell^N}$.
(4) For $v \in Q_N$ $\bar{\rho}$ is unramified at v and is the sum of two representations $\bar{\chi} \oplus \bar{s}$ with $\bar{\chi}$ of dimension 1 and not isomorphic to a subquotient of $\bar{s}$.

Assuming this is possible, then for each N, $R^{univ}_{\mathcal{D}(Q(N))}$ is a power series over $\mathcal{O}$ in $r - n\sum_{v|\infty} \frac{c_v+1}{2}$ variables.

Proposition. *Let* $H = \bar{r}(Gal(\overline{\mathbb{Q}}/F^+(\zeta_\ell))$. *Suppose*

$$h^0(H, ad\ \bar{r}) = h^1(H, ad\ \bar{r}) = 0$$

and for any irreducible $k[H]$*-submodule* W *of* $ad\ \bar{r}$ *there is* $h \in H \cap \bar{r}(\Delta)$ *and* $\alpha \in k$ *such that*

(a) *The* α*-generalized eigenspace* $V_{h,\alpha}$ *of* h *in* k^n *is one-dimensional*
(b) $pr_{V_{h,\alpha}} \circ W \circ i_{V_{h,\alpha}} \subset ad\ \bar{r}$ *is a non-zero subspace.*

Then the sets Q_N *as above exist for all* N.

The conditions on H amount to saying that H is "big". If $ad\ \bar{r}$ is irreducible upon restriction to $H \cap \bar{r}\Delta$, then the last condition is automatic. Verification of these conditions in general is a technical point we will ignore. In some applications, one can vary ℓ so as to guarantee these conditions. The optional hypothesis will be assumed (implicitly, without calling attention to itself) where it simplifies the discussion.

Henceforward, we consider the conditions at S given, and change the notation: we let

$$R_{\bar{r},\emptyset} = R_{\mathcal{D}}^{univ}; \quad R_{\bar{r},Q} = R_{\mathcal{D}(Q)}^{univ}.$$

This requires specifying conditions $(\mathcal{D}_v, L_v)$ for $v \in Q$ so that $\dim_k L_v - h^0(\Delta_v, ad\,\bar{r}) = 1$. The condition is that $\bar{\rho}$ lifts to $\rho \xrightarrow{\sim} \chi \oplus s$ where χ lifts $\bar{\chi}$ but is allowed to ramify but s is still unramified. The condition on the dimension is easy to check.

I have said nothing about automorphic forms today! The objective is to show that every lifting of $\bar{r}$ of the appropriate type at S_ℓ is of the form $r(\Pi)$ for some automorphic representation Π of $GL(n)$ of the kind considered in the first two lectures. The more modest objective is to show that every lifting of $\bar{r}$ that is unramified outside $S \coprod S_\ell$, minimal at S, and of the appropriate type at S_ℓ is of the form $r(\Pi)$. We call these liftings minimal. Now a minimal lifting of $\bar{r}$ to $\mathcal{G}_n(\mathcal{O}')$, where $\mathcal{O}'$ is an ℓ-adic integer ring contained in $\mathcal{C}_{\mathcal{O}}$, corresponds to a homomorphism $R_{\bar{r},\emptyset} \to \mathcal{O}'$. So we need to show that every such homomorphism comes from an automorphic representation. The goal is in fact to show that $R_{\bar{r},\emptyset}$ is isomorphic to an algebra that already has this property – specifically, to the algebra generated by Hecke algebras acting on automorphic forms of a certain type (unramified outside S, for example). However, we have already said enough today.

IV. Deformation rings and Hecke algebras

8. Hecke algebras and the Taylor-Wiles theorem.

9. Level raising.