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# Sheaf Cohomology

Paul Garrett [garrett@math.umn.edu](mailto:garrett@math.umn.edu) <http://www.math.umn.edu/~garrett/>

There is a fundamental tension between the good formal properties of sheaf cohomology defined as right-derived functors of the global-sections functor, and the computational accessibility of Čech cohomology.

Here we prove, under the reasonable assumption of *local acyclicity* of a sheaf, that the two notions of cohomology agree.

The essential auxiliary notion for comparison of Čech and derived-functor cohomology is that of *flasque* sheaf. We also give a resume of other standard adjectives applicable to sheaves.

- Computing by acyclic resolutions
- The global sections functor  $\Gamma(X, *)$
- Conditions for acyclicity: some adjectives
- Basic properties of flasque sheaves
- Ringed spaces
- Injective  $\mathbf{O}_X$ -modules are flasque
- Flasque implies  $\Gamma(X, \cdot)$ -acyclic
- Čech cohomology of sheaves
- Čech-acyclicity of flasque sheaves
- Local acyclicity and comparison of cohomologies

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## 1. Computing by acyclic resolutions

We first review the way that right derived functors are defined, in terms of *injectives*, and extend this in a trivial but essential way, to computation via *acyclic resolutions*.

In the definition of right derived functors  $R^i F$  of a left-exact functor  $F$ , one considers *injective resolutions*

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{I}_0 \xrightarrow{f_0} \mathcal{I}_1 \xrightarrow{f_1} \dots$$

of modules, or of sheaves  $\mathcal{S}$  on topological spaces  $X$ , or whatever, applies  $F$  to obtain

$$0 \rightarrow F\mathcal{S} \rightarrow F\mathcal{I}_0 \xrightarrow{Ff_0} F\mathcal{I}_1 \xrightarrow{Ff_1} \dots$$

and then takes the ‘deleted complex’

$$0 \rightarrow F\mathcal{I}_0 \xrightarrow{Ff_0} F\mathcal{I}_1 \xrightarrow{Ff_1} \dots$$

and then takes cohomology of the latter complex:

$$R^n F(\mathcal{S}) := \ker Ff_n / \operatorname{im} Ff_{n-1}$$

This procedure defines the right derived functors  $R^n F$  of  $F$ .

One then proves that, given two different injective resolutions, there is a *chain homotopy* between them, so that the cohomology of the two image complexes is the same, so *the definition of right derived functor is independent of injective resolution*.

More generally, for a fixed functor  $F$ , say that an object  $\mathcal{A}$  is  $(F)$ -**acyclic** if all higher right derived functors of  $F$  annihilate  $\mathcal{A}$ , i.e., if  $R^n F(\mathcal{A}) = 0$  for  $n > 0$ .

In fact, the elementary homological algebra proof that right derived functors' definitions do not depend upon the choice of injective resolution shows that, if

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{I}_0 \xrightarrow{f_0} \mathcal{I}_1 \xrightarrow{f_1} \dots$$

is injective and

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{A}_0 \xrightarrow{g_0} \mathcal{A}_1 \xrightarrow{g_1} \dots$$

is merely *F-acyclic*, then we have a natural *chain homotopy* from

$$0 \rightarrow F\mathcal{S} \rightarrow F\mathcal{I}_0 \xrightarrow{Ff_0} F\mathcal{I}_1 \xrightarrow{Ff_1} \dots$$

to

$$0 \rightarrow F\mathcal{S} \rightarrow F\mathcal{A}_0 \xrightarrow{Fg_0} F\mathcal{A}_1 \xrightarrow{Fg_1} \dots$$

Therefore, we have a natural

$$R^n F(\mathcal{S}) \approx \ker Fg_n / \operatorname{im} Fg_{n-1}$$

That is, *if there is at least one injective resolution of an object  $\mathcal{S}$ , then the derived functors  $R^n F$  of  $F$  can be computed via any  $F$ -acyclic resolution.*

## 2. The global-sections functor $\Gamma(X, *)$

The right-derived functor procedure yields (**'true'**) **sheaf cohomology** if  $F$  is the **global sections functor**

$$F(\mathcal{S}) = \Gamma(X, \mathcal{S}) := \text{sections of } \mathcal{S} \text{ on all of } X$$

That is, *sheaf cohomology is an example of a right derived functor.*

Here we note that the global-sections functor  $\Gamma(X, *)$  is *left-exact*.

## 3. Some adjectives

The very notion of 'injective' is 'extrinsic' to the extent that it depends upon the category. Further, from the definition of right derived functors it follows immediately that injective sheaves are acyclic for *any* right derived functors. One certainly might imagine that this 'universal acyclicity' of injectives is more than what is needed for the global sections functor alone.

For the particular functor 'take global sections', other conditions on a sheaf still guarantee acyclicity and at the same time *are 'intrinsic'* in that they do not refer to any 'ambient category'. Therefore, in principle these other conditions are more readily verifiable.

For the moment, we merely catalogue the most common adjectives potentially applicable to sheaves. In many cases one proves that such sheaves are acyclic, but this is not to be taken for granted.

A sheaf is **flabby** (=Fr. 'flasque') or **scattered** if sections on *open* subsets always extend to the whole space.

A sheaf is **soft** (=Fr. 'mou') if sections on *closed* subsets always extend to the whole space.

A sheaf is **c-soft** if sections on compact subsets always extend to the whole space.

A sheaf  $\mathcal{S}$  is **fine** if, for every locally finite covering of the space by opens  $U_\alpha$ , there is a collection of endomorphisms  $\phi_\alpha$  of the sheaf so that  $\phi_\alpha s_x = 0$  if  $x \notin U_\alpha$ , and so that  $\sum_\alpha \phi_\alpha = \operatorname{id}_{\mathcal{S}}$ .

## 4. Basic properties of flasque sheaves

**Lemma:** Products of flasque sheaves are flasque. ///

For a continuous map  $f : X \rightarrow Y$ , recall that the **direct image functor**  $f_*$  mapping sheaves on  $X$  to sheaves on  $Y$  is defined by

$$(f_*\mathcal{S})(U) = \mathcal{F}(f^{-1}U)$$

for an open set  $U$  in  $Y$ . The image  $f_*\mathcal{S}$  is the **direct image sheaf**.

Let  $j : U \rightarrow X$  be an inclusion with  $U$  open in  $X$ . Let  $\mathcal{S}$  be a sheaf on  $U$ . Define a *presheaf* by sending an open  $V$  to  $\mathcal{S}(V)$  if  $V \subset U$  and sending  $V$  to  $\{0\}$  otherwise. Let  $j_!\mathcal{S}$  be the *sheaf* obtained from this presheaf. Then  $j_!\mathcal{S}$  is the **extension by zero** of  $\mathcal{S}$  from the *open* subset  $U$  to  $X$ . (The symbol  $j_!$  is pronounced ‘j-lower-shriek’).

**Lemma:** If  $i : Z \rightarrow X$  is the inclusion of a *closed* subset, and  $\mathcal{S}$  is a sheaf on  $Z$ , show that  $i_*\mathcal{S}$  has the property that its stalks agree with those of  $\mathcal{S}$  on  $Z$  and are zero outside  $Z$ . (Thus, this is the extension-by-zero from a *closed* subset).

*Proof:*

**Lemma:** If  $j : U \rightarrow X$  is the inclusion of an *open* subset, and  $\mathcal{S}$  is a sheaf on  $U$ , show that  $j_!\mathcal{S}$  is the unique sheaf on  $X$  with the property that its stalks agree with those of  $\mathcal{S}$  on  $U$  and are zero outside  $Z$ , and whose *restriction* to  $U$  is again  $\mathcal{S}$ .

**Lemma:** Let  $j : U \rightarrow X$  be an inclusion of an open  $U$ , and let  $i : Z \rightarrow X$  the the inclusion of the closed complement  $Z = X - U$ . Show that

$$0 \rightarrow j_!(\mathcal{S}|_U) \rightarrow \mathcal{S} \rightarrow i_*(\mathcal{S}|_Z) \rightarrow 0$$

*Proof:*

**Lemma:** If  $j : X \rightarrow Y$  is continuous and if  $\mathcal{F}$  is a flasque sheaf on  $X$ , then  $j_*\mathcal{F}$  is a flasque sheaf on  $Y$ .

*Proof:*

**Lemma:** Let

$$0 \rightarrow \mathcal{F} \xrightarrow{f} \mathcal{S} \xrightarrow{g} \mathcal{Q} \rightarrow 0$$

be an exact sequence of sheaves of abelian groups on a topological space  $X$ . If  $\mathcal{F}$  is flasque then there is a short exact sequence

$$0 \rightarrow \Gamma(X, \mathcal{F}) \xrightarrow{\Gamma f} \Gamma(X, \mathcal{S}) \xrightarrow{\Gamma g} \Gamma(X, \mathcal{Q}) \rightarrow 0$$

If  $\mathcal{S}$  is flasque, then  $\mathcal{Q}$  is flasque.

*Proof:* The exactness of

$$0 \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{S}) \rightarrow \Gamma(X, \mathcal{Q})$$

is elementary, and holds for any sheaves. For the surjectivity of  $\Gamma(X, \mathcal{S}) \rightarrow \Gamma(X, \mathcal{Q})$ , let  $\sigma$  be a section of  $\mathcal{Q}$  on  $X$ . Stalkwise at  $x \in X$  there is  $s_x \in \mathcal{S}_x$  mapping to  $\sigma_x$ , since  $\mathcal{S} \rightarrow \mathcal{Q}$  is onto. Then consider the family of pairs  $(U, s)$  of open sets  $U$  and sections  $s \in \Gamma(U, \mathcal{S})$  so that

$$(\Gamma g)(s) = \sigma|_U$$

Since we have stalkwise surjectivity, this collection is non-empty. As usual, via Zorn’s lemma we can conclude that there are maximal elements in this family, ordered by  $(U, s) \leq (U', s')$  if  $U \subset U'$  and  $s'|_U = s$ . Let  $(U, s)$  be maximal.

Suppose that  $U$  is strictly smaller than  $X$ . Take  $x \in X - U$  and a sufficiently small neighborhood  $V$  of  $x$  with  $t \in \Gamma(V, \mathcal{S})$  so that  $(\Gamma g)t = \sigma|_V$ . Then

$$(\Gamma g)(s|_{U \cap V} - t|_{U \cap V}) = 0 \in \Gamma(U \cap V, \mathcal{S})$$

so by the exactness of the sequence

$$o \rightarrow \Gamma(U \cap V, \mathcal{F}) \rightarrow \Gamma(U \cap V, \mathcal{S})$$

there is  $\phi \in \Gamma(U \cap V, \mathcal{F})$  which maps to  $s|_{U \cap V} - t|_{U \cap V}$ . Invoking the flasqueness of  $\mathcal{F}$ ,  $\phi$  extends to an element  $\Phi \in \Gamma(X, \mathcal{F})$ . Then we can extend  $s$  to a section of  $\mathcal{S}$  on  $U \cup V$  by taking  $t - (\Gamma g)\Phi|_V$  on  $V$ . This contradicts the maximality of  $U$ , and thereby proves that  $\Gamma g$  is onto.

Now suppose that  $\mathcal{S}$  is flasque. Let  $\sigma \in \Gamma(U, \mathcal{Q})$  for some non-empty open  $U$  in  $X$ . Note that a flasque sheaf restricted to a subset is still flasque. Then the result just proven shows that  $\Gamma(U, \mathcal{S}) \rightarrow \Gamma(U, \mathcal{Q})$  is onto. Take  $s \in \Gamma(U, \mathcal{S})$  mapping to  $\sigma$ . Since  $\mathcal{S}$  is flasque,  $s$  extends to an element of  $\Gamma(X, \mathcal{S})$ , which maps by  $\Gamma g$  to an extension of  $\sigma$  to an element of  $\Gamma(X, \mathcal{Q})$ . Thus,  $\mathcal{Q}$  is flasque. ///

## 5. Ringed spaces

Recall the definition: a **ringed space**  $(X, \mathbf{O}_X)$  is a topological space  $X$  with a sheaf of *rings*  $\mathbf{O} = \mathbf{O}_X$  on it. The latter sheaf of rings is the **structure sheaf** of  $X$ . Smooth manifolds, complex manifolds, and topological manifolds can be viewed as such.

An  **$\mathbf{O}_X$ -module** is a sheaf  $\mathcal{M}$  of abelian groups so that, for all opens  $U$ ,  $\mathcal{M}(U)$  is an  $\mathbf{O}(U)$ -module, and in a manner ‘compatible with all the restriction maps’. The compatibility requirement is that, if

$$\rho_{UV}^{\mathbf{O}} : \mathbf{O}(U) \rightarrow \mathbf{O}(V)$$

is the restriction map for  $\mathbf{O}$  and

$$\rho_{UV}^{\mathcal{M}} : \mathcal{M}(U) \rightarrow \mathcal{M}(V)$$

is the restriction mapping for  $\mathcal{M}$ , (with  $U \supset V$ ) then the compatibility requirement on the module-structures is that, for  $r \in \mathbf{O}(U)$  and  $m \in \mathcal{M}(U)$ ,

$$\rho_{UV}^{\mathcal{M}}(rm) = (\rho_{UV}^{\mathbf{O}}r)(\rho_{UV}^{\mathcal{M}}m)$$

Sheaves of abelian groups may naturally be viewed as  $\mathbf{O}$ -modules for  $\mathbf{O}$  being the sheaf of germs of locally constant  $\mathbf{Z}$ -valued functions.

Note that injectivity in the category of  $\mathbf{O}$ -modules is possibly a weaker requirement than injectivity in some *larger* category of sheaves.

## 6. Injective modules are flasque

The assertion of this section title is stronger than the (true) assertion that injective sheaves of abelian groups are flasque: the category of  $\mathbf{O}_X$ -modules is smaller than that of injective sheaves of abelian groups, and therefore the condition of injectivity as  $\mathbf{O}_X$ -module is *weaker*. Further, in some cases we will be given injectivity as  $\mathbf{O}_X$ -module, rather than injectivity as sheaf of abelian groups, as a natural hypothesis, so we would need the more general and stronger assertion.

**Proposition:** Any sheaf  $\mathcal{M}$  on a ringed space  $(X, \mathbf{O}_X)$  which is an  $\mathbf{O}_X$ -module and which is injective in the category of  $\mathbf{O}_X$ -modules is flasque.

*Proof:* Let  $\mathbf{O}_U$  be the structure sheaf restricted to an open  $U$  in  $X$  and extended by 0 outside  $U$ . Let  $\mathcal{I}$  be an injective  $\mathbf{O}$ -module, and let  $V \subset U$  be opens. Then we have an inclusion  $\mathbf{O}_V \rightarrow \mathbf{O}_U$  of  $\mathbf{O}$ -modules. The definition of injectivity of  $\mathcal{I}$  yields the surjectivity of the natural map

$$\mathrm{Hom}(\mathbf{O}_U, \mathcal{I}) \rightarrow \mathrm{Hom}(\mathbf{O}_V, \mathcal{I})$$

Since we have natural identifications

$$\mathrm{Hom}(\mathbf{O}_U, \mathcal{I}) \approx \Gamma(U, \mathcal{I})$$

we can conclude that  $\mathcal{I}$  is indeed flasque. ///

## 7. Flasque implies acyclic

Let  $(X, \mathbf{O})$  be a ringed space. We may consider any topological space  $X$  as a ringed space, by taking  $\mathbf{O}$  to be the sheaf of germs of locally constant  $\mathbf{Z}$ -valued functions, for example. *We grant that every  $\mathbf{O}$ -module on  $X$  may be imbedded into an injective  $\mathbf{O}$ -module.* That is, we grant that *in the category of  $\mathbf{O}$ -modules there are sufficiently many injective.*

**Proposition:** A flasque  $\mathbf{O}$ -module  $\mathcal{F}$  on  $X$  is acyclic for the global sections functor  $\Gamma(X, \cdot)$ . That is, for  $i > 0$  we have  $H^i(X, \mathcal{F}) = 0$

*Proof:* Let  $\mathcal{F} \rightarrow \mathcal{I}$  be an imbedding into an injective  $\mathbf{O}$ -module  $\mathcal{I}$ , and let  $\mathcal{Q}$  be the quotient. From above, a quotient of flasque sheaves is flasque. Therefore,  $\mathcal{Q}$  is flasque. Also, we have shown that we have an exact sequence

$$0 \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{I}) \rightarrow \Gamma(X, \mathcal{Q}) \rightarrow 0$$

Since  $\mathcal{I}$  is injective, it is acyclic. Thus, from the long exact cohomology sequence we get

$$H^1(X, \mathcal{F}) = 0$$

$$H^i(X, \mathcal{F}) \approx H^{i-1}(X, \mathcal{Q})$$

for  $i > 1$ . By induction on  $i$ , we get the result. ///

## 8. Čech cohomology of sheaves

Let  $\mathcal{U}$  be an open cover of a topological space  $X$ . Define an **oriented  $q$ -simplex** with respect to  $\mathcal{U}$  to be an ordered  $(q+1)$ -tuple  $\sigma = (U_0, \dots, U_q)$  of elements of  $\mathcal{U}$  so that

$$U_0 \cap \dots \cap U_q \neq \emptyset$$

The **support**  $|\sigma|$  of  $\sigma$  is

$$|\sigma| := U_0 \cap \dots \cap U_q$$

Given a sheaf (or even a presheaf)  $\mathcal{S}$  on  $X$ , a **Čech  $q$ -cochain**  $f$  (with coefficients in  $\mathcal{S}$ ) is an assignment

$$(U_p, \dots, U_q) \rightarrow f((U_p, \dots, U_q)) \in \Gamma(U_p \cap \dots \cap U_q, \mathcal{S})$$

for every oriented  $q$ -simplex  $\sigma$  of  $\mathcal{U}$ , so that, further, for a permutation  $\pi$  of indices  $0, \dots, q$  we have

$$f(U_{\pi 0}, \dots, U_{\pi q}) = \text{sign}(\pi) f(U_p, \dots, U_q)$$

where  $\text{sign}(\pi)$  is the sign of the permutation.

The abelian group of all Čech  $q$ -cochains is denoted  $C^q(\mathcal{U}, \mathcal{S})$ .

Note that if some  $U \in \mathcal{U}$  occurs more than once among  $U_0, \dots, U_q$ , then

$$f(U_0, \dots, U_q) = 0$$

Given an oriented  $q$ -simplex  $\sigma$  of  $\mathcal{U}$  we define the (Čech) **coboundary map**

$$\delta : C^q(\mathcal{U}, \mathcal{S}) \rightarrow C^{q+1}(\mathcal{U}, \mathcal{S})$$

by

$$(\delta m)(\sigma) := \sum_{i=0}^{q+1} (-1)^i m(\sigma_i)|_\sigma$$

where with

$$\sigma_i := (U_o, U_1, \dots, \hat{U}_i, \dots, U_{q+1})$$

where the notation  $\hat{U}_i$  indicates as usual that  $U_i$  is *omitted*.

**Lemma:** The Čech coboundary map  $\delta$  satisfies  $\delta^2 = 0$ , so the Čech complex

$$0 \rightarrow C^0(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} \dots$$

associated to the sheaf  $\mathcal{S}$  and the covering  $\mathcal{U}$  really *is* a complex of abelian groups.

*Proof:* The fact that  $\delta^2 = 0$  is a matter of design in the definition, and is easy to check. ///

Then we can define the **Čech cohomology groups** simply as the cohomology of this complex:

$$\check{H}^i(\mathcal{U}, \mathcal{S}) = \ker(\delta \text{ on } C^i(\mathcal{U}, \mathcal{S})) / \delta C^{i-1}(\mathcal{U}, \mathcal{S})$$

The extreme case is

$$\check{H}^0(\mathcal{U}, \mathcal{S}) = \ker(\delta \text{ on } C^0(\mathcal{U}, \mathcal{S}))$$

One essential feature of the Čech cohomology groups is:

**Proposition:** For any sheaf  $\mathcal{S}$ , topological space  $X$ , and covering  $\mathcal{U}$  of  $X$ , the 0<sup>th</sup> Čech cohomology group  $\check{H}^0(\mathcal{U}, \mathcal{S})$  is the group  $\Gamma(X, \mathcal{S})$  of global sections:

$$\check{H}^0(\mathcal{U}, \mathcal{S}) = \Gamma(X, \mathcal{S})$$

*Proof:* Again, we suppose that  $\mathcal{U}$  is *ordered*. If  $f \in C^0(\mathcal{U}, \mathcal{S})$ , then for  $U, V$  in  $\mathcal{U}$  we have

$$(df)(U, V) = f(V)|_{U \cap V} - f(U)|_{U \cap V}$$

or, more tersely,

$$(df)(U, V) = f(V) - f(U)$$

The condition  $df = 0$  would assert that the sections  $f(U) \in \Gamma(U, \mathcal{S})$  and  $f(V) \in \Gamma(V, \mathcal{S})$  have the same image in  $\Gamma(U \cap V, \mathcal{S})$ . From the axioms for a sheaf it follows that all these local sections can be ‘glued together’ to give a global section. ///

To do computations in terms of derived-functor sheaf cohomology, we must ‘sheafify’ the Čech business, which we do now. Among other things, this sheafification engenders some exactness properties otherwise not visible. For a sheaf  $\mathcal{S}$  and open cover  $\mathcal{U}$  define

$$\mathcal{C}^i(\mathcal{U}, \mathcal{S}) = \prod_{U_o \cap \dots \cap U_i} j_* (\mathcal{S}|_{U_o \cap \dots \cap U_i})$$

where the product is taken over  $i + 1$ -fold non-empty intersections

$$U = U_o \cap \dots \cap U_i$$

of distinct elements of  $\mathcal{U}$ , and for each such  $U$  the map  $j : U \rightarrow X$  is the inclusion. We define the sheafified coboundary map by the same formula as before, but with this somewhat different meaning: for  $\sigma \in \mathcal{C}^i(\mathcal{U}, \mathcal{S})$  we put

$$(\delta f)(\sigma) := \sum_{i=0}^{i+1} (-1)^i f(\sigma_i)|_\sigma$$

Again it is easy to check that  $\delta^2 = 0$ . Let  $\mathcal{C}(\mathcal{U}, \mathcal{S})$  denote the complex

$$\mathcal{C}^0(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} \mathcal{C}^1(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} \mathcal{C}^2(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} \mathcal{C}^3(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} \dots$$

From the construction, and from the sheaf axioms, we have

$$\Gamma(X, \mathcal{C}^q(\mathcal{U}, \mathcal{S})) = \mathcal{C}^q(\mathcal{U}, \mathcal{S})$$

**Proposition:** For any sheaf  $\mathcal{S}$  of abelian groups on  $X$ , and for any open cover  $\mathcal{U}$  of  $X$ , there is a natural map  $\epsilon : \mathcal{S} \rightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{S})$  so that  $\mathcal{C}(\mathcal{U}, \mathcal{S})$  is a resolution of  $\mathcal{S}$ , that is, so that we have an exact sequence

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} \mathcal{C}^1(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} \dots$$

*Proof:* Define

$$\epsilon : \mathcal{S} \rightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{S})$$

by taking the product of the natural maps

$$\mathcal{S} \rightarrow j_*(\mathcal{S}|_U)$$

for an open  $J : U \rightarrow X$  in the covering  $\mathcal{U}$ . Then, as in the previous proposition, the sheaf axioms give exactness at the first joint.

To show exactness at the higher joints, by the sheaf axioms it suffices to show exactness *stalk-wise*. As usual, for a sheaf  $\mathcal{S}$  the stalk at  $x$  is denoted by  $\mathcal{S}_x$ . Fix  $x \in X$  fix  $U \in \mathcal{U}$  so that  $x \in U$ . For  $i \geq 1$  we will define a map

$$\theta : \mathcal{C}^i(\mathcal{U}, \mathcal{S})_x \rightarrow \mathcal{C}^{i-1}(\mathcal{U}, \mathcal{S})_x$$

below which will have the property

$$\delta\theta + \theta\delta = 1 \quad \text{on } \mathcal{C}^q(\mathcal{U}, \mathcal{S})_x$$

from which it will follow that the higher joints are exact. (This  $\theta$  is a fragment of a chain homotopy).

**Remarks:** Varying the choice of  $U$  *does* affect  $\theta$ .

We construct  $\theta$  as follows, depending upon choice of  $U$ . For  $f_x \in \mathcal{C}^i(\mathcal{U}, \mathcal{S})_x$ , choose a section  $f$  of this sheaf on some neighborhood  $V$  of  $x$  which maps to  $f_x$  in the direct limit defining the stalk. We may shrink  $V$  if necessary so that  $V \subset U$ . For  $U_o, \dots, U_{i-1}$  all in  $\mathcal{U}$ , define

$$(\tilde{\theta}f)(U_o, \dots, U_{i-1}) = f(U, U_o, \dots, U_{i-1})$$

Then define  $\theta f$  to be the image of  $\tilde{\theta}f$  in the stalk at  $x$ . This does make sense since

$$V \cap U \cap U_o \cap \dots \cap U_{i-1} = V \cap U_o \cap \dots \cap U_{i-1}$$

and  $x \in V \subset U$ .

The key point is the chain homotopy property

$$(\delta\theta + \theta\delta)f = f$$

for  $f \in \mathcal{C}^i(\mathcal{U}, \mathcal{S})_x$  with  $i \geq 1$ . To see this, we compute

$$\begin{aligned} (\tilde{\theta}\delta f)(U_o, \dots, U_i) &= (\delta f)(U, U_o, \dots, U_i) = \\ &= f(U_o, \dots, U_i) - \sum_j (-1)^j f(U, U_o, \dots, \hat{U}_j, \dots, U_i) = \end{aligned}$$

$$= f(U_o, \dots, U_i) - (\delta\theta f)(U_o, \dots, U_i)$$

as desired.

///

## 9. Čech-acyclicity of flasque sheaves

**Proposition:** Let  $\mathcal{F}$  be a flasque sheaf of abelian groups on a topological space  $X$ . Then  $\mathcal{F}$  is Čech-acyclic: for any open cover  $\mathcal{U}$ , for all  $n > 0$  we have  $\check{H}^n(X, \mathcal{F}) = 0$ .

*Proof:* We use the Čech resolution

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \rightarrow \dots$$

We claim that for  $\mathcal{F}$  flasque all the sheaves  $\mathcal{C}^n(\mathcal{U}, \mathcal{F})$  are also flasque. To see this, first observe that for any  $U_o, \dots, U_n$  in  $\mathcal{U}$  with non-empty intersection  $U$  the restriction  $\mathcal{F}|_U$  is still flasque. Let  $j : U \rightarrow X$  be the inclusion. The image of a flasque sheaf under  $j_*$  is flasque, as is the product of flasque sheaves. Thus, from the definition of these Čech sheaves, they are flasque if  $\mathcal{F}$  is flasque. This is the claim.

Then the Čech resolution is *acyclic*, so can be used to compute the cohomology of  $\mathcal{F}$ :

$$H^n(X, \mathcal{F}) = \frac{\ker \delta \text{ on } \Gamma(X, \mathcal{C}^n(\mathcal{U}, \mathcal{F}))}{\delta\Gamma(X, \mathcal{C}^{n-1}(\mathcal{U}, \mathcal{F}))}$$

from the fact that  $\mathcal{F}$  is a sheaf. On the other hand, from the definition of the Čech cohomology groups,

$$\check{H}^n(\mathcal{U}, \mathcal{F}) = \frac{\ker \delta \text{ on } \Gamma(X, \mathcal{C}^n(\mathcal{U}, \mathcal{F}))}{\delta\Gamma(X, \mathcal{C}^{n-1}(\mathcal{U}, \mathcal{F}))}$$

But  $\mathcal{F}$  itself is also flasque and hence acyclic, so  $H^n(X, \mathcal{F}) = 0$ . This shows that the Čech groups  $\check{H}^n(\mathcal{U}, \mathcal{F})$  are zero in positive degrees, also. ///

## 10. Acyclic covers and comparison of cohomologies

The result here is due to Leray. It asserts that, granting ‘acyclicity’ of an open cover with regard to a given sheaf, Čech cohomology (with respect to that cover) and right-derived functor cohomology are the same. Note that the first proposition here holds without any assumption that the sheaf in question be flasque.

The acyclicity of the cover gives us a long exact sequence of the Čech groups, allowing an induction on degree comparing Čech with sheaf cohomology. This induction is an example of what is called ‘dimension-shifting’.

**Proposition:** Let  $X$  be a topological space,  $\mathcal{U}$  an open covering of  $X$ , and  $\mathcal{S}$  a sheaf on  $X$ . Then for each  $n \geq 0$  there is a natural map

$$\check{H}^n(\mathcal{U}, \mathcal{S}) \rightarrow H^n(X, \mathcal{S})$$

which is functorial in  $\mathcal{S}$ .

*Proof:* Consider the situation

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{S} & \rightarrow & \mathcal{C}^0(\mathcal{U}, \mathcal{S}) & \rightarrow & \mathcal{C}^1(\mathcal{U}, \mathcal{S}) & \rightarrow & \dots \\ & & \downarrow & & & & & & \\ 0 & \rightarrow & \mathcal{S} & \rightarrow & \mathcal{I}^0 & \rightarrow & \mathcal{I}^1 & \rightarrow & \dots \end{array}$$



where the vertical arrow is the identity map on  $\mathcal{S}$ , where the lower line is an injective resolution of  $\mathcal{S}$ , and where the upper line is the Čech resolution. Note that we are not asserting anything about acyclicity of the Čech resolution in this situation. From the definition of injective, we can extend this diagram to a diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{S} & \rightarrow & \mathcal{C}^0(\mathcal{U}, \mathcal{S}) & \rightarrow & \mathcal{C}^1(\mathcal{U}, \mathcal{S}) & \rightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathcal{S} & \rightarrow & \mathcal{I}^0 & \rightarrow & \mathcal{I}^1 & \rightarrow & \dots \end{array}$$

where all the squares commute. That is, we have a morphism of complexes (of sheaves). Further, as usual, this extension is unique up to chain homotopy. Then, applying the functor  $\Gamma(X, *)$  and taking (co-)homology of the resulting complexes, we obtain the natural map indicated. ///

Let  $\mathcal{S}$  be a sheaf on a topological space  $X$ . Let  $\mathcal{U}$  be an open cover of  $X$  so that, for every  $U_o, \dots, U_n \in \mathcal{U}$ , and for every  $i > 0$ , we have

$$H^i(U_o \cap \dots \cap U_n, \mathcal{S}|_{U_o \cap \dots \cap U_n}) = 0$$

Then we say that the cover  $\mathcal{U}$  is **acyclic for the sheaf  $\mathcal{S}$** .

**Theorem:** Let  $\mathcal{S}$  be a sheaf on a topological space  $X$ , and let  $\mathcal{U}$  be an open cover of  $X$  which is acyclic for  $\mathcal{S}$ . Then for all  $i \geq 0$  the natural maps

$$\check{H}^i(\mathcal{U}, \mathcal{S}) \rightarrow H^i(X, \mathcal{S})$$

(just above) are *isomorphisms*.

*Proof:* Since the 0<sup>th</sup> Čech cohomology is the group of global sections of  $\mathcal{S}$ , which is the 0<sup>th</sup> right-derived functor of the global-sections functor, we have the assertion for  $i = 0$  with or without the acyclicity of  $\mathcal{U}$  for  $\mathcal{S}$ .

For the induction step, imbed  $\mathcal{S}$  in an injective sheaf  $\mathcal{I}$ , and let  $\mathcal{Q}$  be the quotient sheaf: we have a short exact sequence

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{I} \rightarrow \mathcal{Q} \rightarrow 0$$

For each non-empty  $q + 1$ -fold intersection

$$U = U_o \cap \dots \cap U_q$$

of elements of the cover  $\mathcal{U}$ , we obtain a long exact sequence in cohomology (as right-derived functor)

$$\begin{aligned} 0 \rightarrow H^0(U, \mathcal{S}|_U) &\rightarrow H^0(U, \mathcal{I}|_U) \rightarrow H^0(U, \mathcal{Q}|_U) \rightarrow H^1(U, \mathcal{S}|_U) \rightarrow \dots \\ &\rightarrow H^i(U, \mathcal{S}|_U) \rightarrow H^i(U, \mathcal{I}|_U) \rightarrow H^i(U, \mathcal{Q}|_U) \rightarrow \dots \end{aligned}$$

By the hypothesis of acyclicity of  $\mathcal{U}$  for  $\mathcal{S}$ ,  $H^1(U, \mathcal{S}|_U) = 0$ , so, recalling that  $H^0(U, *) = \Gamma(U, *)$ , we have the short exact sequence

$$0 \rightarrow \Gamma(U, \mathcal{S}) \rightarrow \Gamma(U, \mathcal{I}) \rightarrow \Gamma(U, \mathcal{Q}) \rightarrow 0$$

And from the ‘higher’ parts of the long exact sequence, invoking the acyclicity of  $\mathcal{U}$  for  $\mathcal{S}$  and the acyclicity of injective (hence, flasque) sheaves, we conclude that for  $i > 0$  we have

$$\check{H}^i(U, \mathcal{Q}) = 0$$

That is, *the same cover  $\mathcal{U}$  is also acyclic for the quotient sheaf  $\mathcal{Q}$* .

Taking products of these exact sequences, we obtain an exact sequence of complexes

$$0 \rightarrow C^\cdot(\mathcal{U}, \mathcal{S}) \rightarrow C^\cdot(\mathcal{U}, \mathcal{I}) \rightarrow C^\cdot(\mathcal{U}, \mathcal{Q}) \rightarrow 0$$

Obtaining this short exact sequence of sheaves was the point of the assumption of the acyclicity of the cover for the sheaf  $\mathcal{S}$ . From this we obtain, as usual, by homological algebra, the long exact sequence of the (co-)homologies of the complexes. In this setting these cohomologies are the Čech cohomology groups with respect to the cover  $\mathcal{U}$ :

$$\begin{array}{ccccccc} 0 & \rightarrow & \check{H}^0(\mathcal{U}, \mathcal{S}) & \rightarrow & \check{H}^0(\mathcal{U}, \mathcal{I}) & \rightarrow & \check{H}^0(\mathcal{U}, \mathcal{Q}) & \rightarrow \\ & & \check{H}^1(\mathcal{U}, \mathcal{S}) & \rightarrow & \check{H}^1(\mathcal{U}, \mathcal{I}) & \rightarrow & \check{H}^1(\mathcal{U}, \mathcal{Q}) & \rightarrow \dots \\ & & \check{H}^i(\mathcal{U}, \mathcal{S}) & \rightarrow & \check{H}^i(\mathcal{U}, \mathcal{I}) & \rightarrow & \check{H}^i(\mathcal{U}, \mathcal{Q}) & \rightarrow \dots \end{array}$$

Since  $\mathcal{I}$  is injective it is flasque. Thus,  $\check{H}^i(\mathcal{U}, \mathcal{I}) = 0$  for  $i > 0$ , since Čech cohomology groups of flasque sheaves vanish. Thus, the long exact sequence breaks up into smaller exact sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & \check{H}^0(\mathcal{U}, \mathcal{S}) & \rightarrow & \check{H}^0(\mathcal{U}, \mathcal{Q}) & \rightarrow & \check{H}^1(\mathcal{U}, \mathcal{S}) & \rightarrow 0 \\ & & & & & & & \\ & & & & 0 & \rightarrow & \check{H}^{i-1}(\mathcal{U}, \mathcal{Q}) & \rightarrow \check{H}^i(\mathcal{U}, \mathcal{S}) & \rightarrow 0 \end{array}$$

for  $i > 1$ .

On the other hand, in (derived-functor) sheaf cohomology we obtain analogous exact sequences from the long exact sequence arising from

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{I} \rightarrow \mathcal{Q} \rightarrow 0$$

invoking the injectivity of  $\mathcal{I}$ . Putting these together via the natural maps of the previous proposition, we have a commuting diagram with exact rows:

$$\begin{array}{ccccccccccc} 0 & \rightarrow & \check{H}^0(\mathcal{U}, \mathcal{S}) & \rightarrow & \check{H}^0(\mathcal{U}, \mathcal{I}) & \rightarrow & \check{H}^0(\mathcal{U}, \mathcal{Q}) & \rightarrow & \check{H}^1(\mathcal{U}, \mathcal{S}) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & H^0(X, \mathcal{S}) & \rightarrow & H^0(X, \mathcal{I}) & \rightarrow & H^0(X, \mathcal{Q}) & \rightarrow & H^1(X, \mathcal{S}) & \rightarrow & 0 \end{array}$$

where the vertical arrows are the natural maps from above. Note that the naturality is what assures that *the squares commute*.

By diagram-chasing, this implies that  $\check{H}^1(\mathcal{U}, \mathcal{S}) \approx H^1(X, \mathcal{S})$  by the natural map. Further, we have commutative squares

$$\begin{array}{ccccccc} 0 & \rightarrow & \check{H}^{i-1}(\mathcal{U}, \mathcal{Q}) & \rightarrow & \check{H}^i(\mathcal{U}, \mathcal{S}) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & H^{i-1}(X, \mathcal{Q}) & \rightarrow & H^i(X, \mathcal{S}) & \rightarrow & 0 \end{array}$$

with exact rows, where the vertical arrows are the natural maps.

Since, as we noted above, the same cover  $\mathcal{U}$  is acyclic for the quotient  $\mathcal{Q}$ , by induction we know that  $\check{H}^{i-1}(\mathcal{U}, \mathcal{Q}) = 0$ . Then, by induction, *all* the natural maps of the previous proposition must be isomorphisms of the Čech groups to the derived-functor cohomology groups. ///