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exam 08 solutions

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[exam 08 solutions.1] Let $T$ be a hermitian operator on a finite-dimensional complex vector space $V$ with a positive-definite inner product $\langle \cdot , \cdot \rangle$. Let $P$ be an orthogonal projector to the $\lambda$-eigenspace $V_\lambda$ of $T$. (This means that $P$ is the identity on $V_\lambda$ and is 0 on the orthogonal complement $V_\lambda^\perp$ of $V_\lambda$.) Show that $P \in \mathbb{C}[T]$.

By the spectral theorem for hermitian operators, $T$ is diagonalizable. So its minimal polynomial $f(x) \in \mathbb{C}[x]$ factors into linear factors without any repeats, and $f_\lambda(x) = f(x)/(x - \lambda)$ is relatively prime to $x - \lambda$, and (by the Euclidean-ness of $\mathbb{C}[x]$) there are polynomials $a_\lambda(x)$ such that

$$1 = \sum_\lambda a_\lambda(x) f_\lambda(x)$$

Let $P_\lambda = a_\lambda(T)f_\lambda(T) \in \mathbb{C}[T]$.

As usual, these $P_\lambda$s are mutually orthogonal projectors to the eigenspaces of $T$. Again, the argument is that

$$P_\lambda^2 = (a_\lambda(T)f_\lambda(T)) \left( 1 - \sum_{\mu \neq \lambda} a_\mu(T)f_\mu(T) \right) = P_\lambda - \sum_{\mu \neq \lambda} 0 = P_\lambda$$

because $f(x)$ divides $f_\lambda(x) \cdot f_\mu(x)$ for $\mu \neq \lambda$. This shows that $P_\lambda$ is a projector. Also, for $v \in V_\lambda$,

$$P_\lambda v = \left( 1 - \sum_{\mu \neq \lambda} a_\mu(T)f_\mu(T) \right) (v) = v - \sum_{\mu \neq \lambda} a_\mu(T)(0) = v$$

since for $\mu \neq \lambda$ the polynomial $f_\mu(x)$ has a factor of $x - \lambda$, and $(T - \lambda)v = 0$. Similarly, $P_\mu(v) = 0$ for $\mu \neq \lambda$ and $v \in V_\lambda$. This proves that these are projectors to the respective eigenspaces.

For hermitian $T$, eigenspaces for distinct eigenvalues are orthogonal. Thus, $V_\lambda$ is orthogonal to $\sum_{\mu \neq \lambda} V_\mu$.

Since $\sum_\mu V_\mu = V$, it must be that

$$V_\lambda^\perp = \sum_{\mu \neq \lambda} V_\mu$$

At last we can show that the orthogonal projector $P$ to a given $V_\lambda$ is, in fact, the $P_\lambda$ constructed above. Indeed, $P$ is 0 on $V_\mu$ for $\mu \neq \lambda$, since these are in the orthogonal complement to $V_\lambda$, and $P$ is 1 on $V_\lambda$. Since the direct sum of the eigenspaces is the whole space,

$$v = \sum_\mu P_\mu v$$

Thus

$$P = P \circ 1 = P \circ \sum_\mu P_\mu = \sum_\mu P \circ P_\mu = \sum_{\mu \neq \lambda} 0 \cdot P_\mu + \sum_{\mu = \lambda} P_\mu = P_\lambda$$

Thus, $P = P_\lambda = a_\lambda(T)f_\lambda(T) \in \mathbb{C}[T]$. ///

[exam 08 solutions.2] Let $T$ be a diagonalizable operator on a finite-dimensional vector space $V$ over a field $k$. Show that there is a unique projector $P$ to the $\lambda$-eigenspace $V_\lambda$ of $T$ such that $TP = PT$.

(The argument here repeats part of the previous argument, but worth repeating, with minor variations.)

Since $T$ is diagonalizable, its minimal polynomial $f(x) \in k[x]$ factors into linear factors without any repeats, and $g(x) = f(x)/(x - \lambda)$ is relatively prime to $x - \lambda$, and (by the Euclidean-ness of $k[x]$) there are polynomials $a(x), b(x) \in k[x]$ such that

$$1 = a(x)g(x) + b(x)(x - \lambda)$$

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Let $P_\lambda = a(T)g(T) \in k[T]$, and $Q_\lambda = b(T)(T - \lambda)$.

As usual, $P_\lambda, Q_\lambda$ are mutually orthogonal projectors, in the sense that they are projectors and their product is 0. And $P_\lambda$’s image is the $\lambda$ eigenspace, and $Q_\lambda$’s image is a complementary subspace (in fact, the sum of all other eigenspaces, but we don’t need that). The argument is that

$$P_\lambda^2 = a(T)g(T)(1 - b(T)(T - \lambda)) = P_\lambda - 0 = P_\lambda$$

because $f(x) = g(x) \cdot (x - \lambda)f_\mu(x)$. This shows that $P_\lambda$ is a projector. And

$$Q_\lambda^2 = (1 - P_\lambda)^2 = 1 - 2P_\lambda + P_\lambda^2 = 1 - 2P_\lambda + P_\lambda = 1 - P_\lambda = Q_\lambda$$

so $Q_\lambda$ is a projector. Also $P_\lambda Q_\lambda = P_\lambda(1 - P_\lambda) = P_\lambda - P_\lambda^2 = 0$

For $v \in V_\lambda$,

$$P_\lambda v = (1 - b(T)(T - \lambda))(v) = v - 0 = v$$

since $(T - \lambda)v = 0$. And

$$Q_\lambda(v) = (1 - P_\lambda)(v) = 0$$

Finally, for any $v \in V$,

$$v = 1 \cdot v = (P_\lambda + Q_\lambda)v = P_\lambda v + Q_\lambda v$$

so $V$ is the direct sum of the image of $P_\lambda$ and the image of $Q_\lambda$.

The given projector $P$ commutes with $T$, so commutes with $P_\lambda$ and $Q_\lambda$, since these are in $k[T]$. Thus, $P$ stabilizes the 1-eigenspace of $P_\lambda$ and stabilizes the 0-eigenspace for $P_\lambda$ (the latter being the same as the 1-eigenspace of $Q_\lambda$). Thus,

$$P = P \circ 1 = P \circ (P_\lambda + Q_\lambda) = P \circ P_\lambda + P \circ Q_\lambda = 1 \cdot P_\lambda + 0 \cdot Q_\lambda = P_\lambda$$

as desired. //

From just the most basic properties of determinants of matrices, show that the determinant of an upper-triangular matrix is the product of its diagonal entries. That is, show that

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \ldots & a_{nn} \end{pmatrix} = a_{11}a_{22}a_{33}\cdots a_{nn}$$

Let $A$ be the given matrix, and let $a_{ij}$ denote the $ij$th entry, whether or not it’s 0. In the formula

$$\det A = \sum_{\pi \in S_n} \sigma(\pi)a_{\pi(1),1}a_{\pi(2),2}\cdots a_{\pi(n),n}$$

(where $\pi$ runs over the symmetric group $S_n$, and $\sigma$ is the sign function on permutations), note that $a_{\pi(i),i} = 0$ unless $\pi(i) \leq i$. Thus, the $\pi$th summand is 0 unless $\pi(i) \leq i$ for all $i$. Thus, $\pi(1) = 1$. By induction, $\pi(i) = i$ for all $i$. Thus, only the identity permutation appears (with non-zero summand) in the sum, which yields the asserted formula. //