Albert Girard (1629), and, later, Isaac Newton (1666), expressed the elementary symmetric functions $s_j = \sum_{i_1 < i_2 < \ldots < i_j} x_{i_1} x_{i_2} \ldots x_{i_j}$
in terms of symmetric power sum functions $p_j = x_1^j + \ldots + x_n^j$
where $x_1, \ldots, x_n$ are indeterminates.

Basic properties of $\exp$ and $\log$, either as convergent or formal power series, produce the relation. Thus, consider

$$\prod_i (1 - zx_i) = \exp \sum_i \log(1 - zx_i)$$

with an indeterminate $z$, and evaluate this in the two obvious ways. First, of course, the left-hand side essentially defines the elementary symmetric functions:

$$\prod_i (1 - zx_i) = \sum_j (-1)^j s_j z^j$$

On the right-hand side, use the power series for $\log$, interchange order of summation, use the fact that $\exp$ converts sums to products, and expand $\exp$: this will inevitably produce the relation. Indeed, the point is not the specific formula, but the device by which to recover it. We have

$$\exp \left( - \sum_{i} \sum_{n \geq 1} \frac{z^n x_i^n}{n} \right) = \exp \left( - \sum_{k \geq 1} \frac{z^k p_k}{k} \right) = \prod_{k \geq 1} \exp \left( - \frac{z^k p_k}{k} \right) = \prod_{k \geq 1} \sum_{\ell \geq 0} \frac{(-z^k p_k/\ell)!}{\ell!} \ldots$$

Equating the coefficients of $z^j$ in the latter and in $\sum_j (-1)^j s_j z^j$ expresses the elementary symmetric function $s_j$ in terms of sums-of-powers $p_j$:

$$(-1)^j s_j = \sum_{\ell_1 + 2\ell_2 + \ldots + j \ell_j = j} \frac{(-p_1/1)^{\ell_1} (-p_2/2)^{\ell_2} (-p_3/3)^{\ell_3} \ldots (-p_n/n)^{\ell_n}}{\ell_1! \ell_2! \ell_3! \ldots \ell_n!} \ldots$$

Since $\ell_i \geq 1$, the right-hand side of the latter is smaller than it might otherwise appear, namely, the formula for $s_j$ it terminates at the $j^{th}$ term:

$$(-1)^j s_j = \sum_{\ell_1 + 2\ell_2 + \ldots + j \ell_j = j} \frac{(-p_1/1)^{\ell_1} (-p_2/2)^{\ell_2} (-p_3/3)^{\ell_3} \ldots (-p_j/j)^{\ell_j}}{\ell_1! \ell_2! \ell_3! \ldots \ell_j!} \ldots$$

This expresses the elementary symmetric functions in terms of the symmetric power sums. Note that their is a clear limitation on the integers appearing in denominators.

In the opposite direction, while we already know on general principles that the symmetric power sums are expressible in terms of the elementary symmetric functions, a variant of the above argument gives a formulaic expression, as follows. Again, the point is the device by which to recover the formula, not the formula itself.

From the intermediate result (above)

\[
\sum_{0 \leq j \leq n} (-1)^j s_j z^j = \prod_i (1 - zx_i) = \exp \left( -\sum_{k \geq 1} \frac{z^k}{k} p_k \right)
\]

move the \( \exp \) to the left-hand side, as a logarithm:

\[
\log \left( \sum_{0 \leq j \leq n} (-1)^j s_j z^j \right) = -\sum_{k \geq 1} \frac{z^k}{k} p_k
\]

Moving the sign to the other side,

\[
-\log \left( 1 - \sum_{1 \leq j \leq n} (-1)^{j-1} s_j z^j \right) = \sum_{k \geq 1} \frac{z^k}{k} p_k
\]

Expand the logarithm on the left-hand side:

\[
\sum_{\ell \geq 1} \left( s_1 z - s_2 z^2 + \ldots + (-1)^{n-1} s_n z^n \right)^\ell / \ell
\]

\[
= \sum_{\ell \geq 1} \sum_{k_1 + 2k_2 + \ldots + nk_n = \ell} z^{k_1 + 2k_2 + \ldots + nk_n} \frac{1}{\ell! \left(k_1 k_2 \ldots k_n\right)} s_1^{k_1} (-s_2)^{k_2} \ldots (-1)^{n-1} s_n^{k_n}
\]

\[
= \sum_{k \geq 1} z^k \sum_{k_1 + 2k_2 + \ldots + nk_n = k} \frac{(k_1 + k_2 + \ldots + k_n - 1)!}{k_1! k_2! \ldots k_n!} s_1^{k_1} (-s_2)^{k_2} \ldots (-1)^{n-1} s_n^{k_n}
\]

Equating coefficients of \( z^k \),

\[
\sum_{k_1 + 2k_2 + \ldots + nk_n = k} \frac{(k_1 + k_2 + \ldots + k_n - 1)!}{k_1! k_2! \ldots k_n!} s_1^{k_1} (-s_2)^{k_2} \ldots (-1)^{n-1} s_n^{k_n} = \frac{p_k}{k}
\]