Our goal is proof that functors

\[ M \to M \otimes X \]

(for example, from \( \mathbb{Z} \)-modules to \( \mathbb{Z} \)-modules) are right exact. Direct proof is non-trivial. The more pleasant argument introduces adjoint functors and proves a simple form of Yoneda’s lemma. The argument illustrates functoriality of isomorphisms.

To reduce complications and lighten the notation, we treat only \( \mathbb{Z} \)-modules (that is, abelian groups). In particular, spaces \( \text{Hom}(A, B) \) are again abelian groups, as are tensor products \( A \otimes B \), so these stay inside the category of \( \mathbb{Z} \)-modules.

- \( M \to \text{Hom}(X, M) \) is left exact
- Adjointness of \( \text{Hom} \) and \( \otimes \)
- Yoneda lemma
- Half-exactness of adjoint functors

1. \( M \to \text{Hom}(X, M) \) is left exact

The proof is straightforward.

\[ 0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0 \text{ exact} \implies 0 \to \text{Hom}(X, A) \xrightarrow{i_{\circ -}} \text{Hom}(X, B) \xrightarrow{q_{\circ -}} \text{Hom}(X, C) \text{ exact} \]

where the induced maps are by composition with \( i \) and with \( q \) as indicated. Similarly, for the other \( \text{Hom} \) functor \( M \to \text{Hom}(M, X) \) attached to \( X \),

\[ 0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0 \text{ exact} \implies 0 \to \text{Hom}(C, X) \xrightarrow{-q_{\circ}} \text{Hom}(B, X) \xrightarrow{-i_{\circ}} \text{Hom}(C, X) \text{ exact} \]

1.0.2 Remark: The Hom functor \( M \to \text{Hom}(X, M) \) is covariant, in the usual sense that a morphism \( f : M \to N \) gives an arrow in the same direction

\[ \text{Hom}(X, M) \xrightarrow{f_{\circ -}} \text{Hom}(X, N) \]

The other Hom functor \( M \to \text{Hom}(M, X) \) is contravariant, in the usual sense that a morphism \( f : M \to N \) gives an arrow in the opposite direction

\[ \text{Hom}(N, X) \xrightarrow{-f_{\circ}} \text{Hom}(M, X) \]

Proof: For \( f \in \text{Hom}(X, A) \), \( i \circ f = 0 \) implies \((i \circ f)(x) = 0\) for all \( x \in X \), and then \( f(x) = 0 \) for all \( x \) since \( i \) is an injection. Thus, \( \text{Hom}(X, A) \to \text{Hom}(X, B) \) is an injection, giving exactness at the left joint.

Since \( q \circ i = 0 \), any \( f \in \text{Hom}(X, A) \) is mapped to \( 0 \in \text{Hom}(X, C) \) by \( f \to q \circ i \circ f \). That is, the image of \( i \circ - \) is contained in the kernel of \( q \circ - \). On the other hand, when \( g \in \text{Hom}(X, B) \) is mapped to \( q \circ g = 0 \) in \( \text{Hom}(X, C) \),

\[ g(X) \subset \ker q = \text{Im} i \]
Since $i$ is injective, it is an isomorphism to its image, so there is an inverse $i^{-1}: i(A) \to A$. Since $g(X) \subset \text{Im } i$ we can define

$$f = i^{-1} \circ g \in \text{Hom}(X, A)$$

Certainly $i \circ f = g$, so the kernel is contained in the image. This gives exactness at the middle joint, and the left exactness. The exactness of the other Hom is similar. \(/ / / \) 

[1.0.3] **Remark:** The functor $M \to \text{Hom}(X, M)$ is *not* right exact. For example, with

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$$

with an integer $n > 1$, with $X = \mathbb{Z}/n$ there is no non-zero map of the torsion abelian group $X$ to the free abelian group $\mathbb{Z}$. Similarly, the (contravariant) functor $M \to \text{Hom}(M, X)$ is not right exact.

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### 2. Adjointness of Hom and $\otimes$

Here we introduce *adjoint functors* and give the principal example, adjointness between Hom functors and tensor product functors. The *functoriality* of the isomorphism is explained, and the importance of this functoriality will be illustrated in proving the right exactness of $A \to A \otimes X$.

The adjointness property is related to *Frobenius reciprocity* and *Shapiro’s Lemma*. Let $R$ and $L$ be two functors from the category of $\mathbb{Z}$-modules to itself. These two functors are *mutually adjoint* when, there is a *functorial* isomorphism

$$\text{Hom}(LA, B) \approx \text{Hom}(A, RB) \quad \text{(for all } A, B)$$

The functor $R$ is a *right adjoint*, and $L$ is a *left adjoint*. *Functoriality* means that, for each pair of morphisms $f: A' \to A$ and $g: B \to B'$ (yes, the maps go from $A'$ to $A$, but from $B$ to $B'$) we have a commutative diagram$[^1]$

$$
\begin{array}{ccc}
\text{Hom}(LA, B) & \approx & \text{Hom}(A, RB) \\
g \circ (\ast) \circ Lf & \downarrow & Rg \circ (\ast) \circ f \\
\text{Hom}(LA', B') & \approx & \text{Hom}(A', RB')
\end{array}
$$

That is,

$$g \circ F \circ Lf = Rg \circ F \circ f \quad \text{(for every } F \in \text{Hom}(LA, B)) \]$$

[2.0.1] **Theorem:** For $\mathbb{Z}$-modules $A, X, B$ we have a *functorial* isomorphism

$$\text{Hom}(A \otimes X, B) \approx \text{Hom}(A, \text{Hom}(X, B))$$

**Proof:** Given $\Phi \in \text{Hom}(A \otimes X, B)$, define $\varphi_\Phi \in \text{Hom}(A, \text{Hom}(X, B))$ by

$$\varphi_\Phi(a)(x) = \Phi(a \otimes x)$$

Conversely, given $\varphi \in \text{Hom}(A, \text{Hom}(X, B))$, define $\Phi_\varphi \in \text{Hom}(A \otimes X, B)$ by

$$\Phi_\varphi(a \otimes x) = \varphi(a)(x)$$

[^1]: Assembling these isomorphisms into larger diagrams is critical in proving the half-exactness results below.
Paul Garrett: Half-exactness of adjoint functors, Yoneda lemma (December 10, 2008)

and extending by linearity. Visibly the maps $\Phi \to \varphi_\Phi$ and $\varphi \to \Phi_\varphi$ are mutual inverses.

The functoriality of the isomorphism refers to the behavior of the isomorphism when we have $f : A' \to A$ and $g : X \to X'$ and/or $h : B \to B'$. (Yes, the order of the primed and unprimed symbols is opposite.) Thus, the diagram

$$\text{Hom}(A \otimes X, B) \approx \text{Hom}(A, \text{Hom}(X, B))$$

$$\Phi \to g \circ \Phi \circ (f \otimes \text{id}_X) \quad \downarrow \quad \downarrow \quad \varphi \to (a' \to g(\varphi(f(a'))(x)))$$

$$\text{Hom}(A' \otimes X, B') \approx \text{Hom}(A', \text{Hom}(X, B'))$$

must commute. This is very easy to check: starting with $\Phi$ in the upper left, going down gives $\Phi \circ (f \otimes \text{id}_X)$, and then going to the right gives $\varphi$ such that

$$\varphi(a')(x) = (\Phi \circ (f \otimes \text{id}_X))(f(a') \otimes x) = \Phi(a \otimes x)$$

Going the other way around the diagram, first we obtain $\varphi$ such that $\varphi(a)(x) = \Phi(a \otimes x)$. Going down the right side gives $\varphi'$ such that

$$\varphi'(a')(x) = \varphi(f(a'))(x) = \Phi(f(a') \otimes x)$$

which is the same as the first computation, so we have the functoriality. ///

3. Yoneda’s lemma

While proving the right exactness of $A \to A \otimes X$ using results above, the following issues arise. This complement to the left-exactness of $M \to \text{Hom}(X, M)$ is a special case of Yoneda’s Lemma. [2]

[3.0.1] Theorem: We have sufficient criteria for exactness:

$$\text{Hom}(X, A) \xrightarrow{f \circ -} \text{Hom}(X, B) \xrightarrow{g \circ -} \text{Hom}(X, C) \quad \text{exact for all } X \implies A \xrightarrow{f} B \xrightarrow{g} C \text{ exact}$$

Also,

$$\text{Hom}(C, X) \xleftarrow{- \circ g} \text{Hom}(B, X) \xleftarrow{- \circ f} \text{Hom}(A, X) \quad \text{exact for all } X \implies A \xrightarrow{f} B \xrightarrow{g} C \text{ exact}$$

[3.0.2] Remark: Exactness of $A \to B \to C$ does not imply exactness of the Hom diagram for all $X$. This was visible in proving left exactness of $M \to \text{Hom}(M, X)$.

Proof: On one hand, with $X = A$ and $F : X \to A$ the identity, exactness of the Hom sequence implies

$$0 = g \circ f \circ F = g \circ f$$

so $\text{Im } f \subset \ker g$. On the other hand, with $X = \ker g$ and $F : X \to B$ the inclusion, exactness of the Hom sequence (with $g \circ F = 0$) implies that there is $F' : X \to A$ such that $f \circ F' = F$. Then

$$\ker g = \text{Im } F = \text{Im } (f \circ F') \subset \text{Im } f$$

Putting the two containments together gives $\ker g = \text{Im } f$. This proves the result for the covariant Hom functor.

[2] Such a map $X \to \text{Hom}(X, A)$ of objects, from a category whose sets $\text{Hom}(A, B)$ of maps are abelian groups, to the category of abelian groups, is called a Yoneda imbedding.
Paul Garrett: Half-exactness of adjoint functors, Yoneda lemma (December 10, 2008)

For the contravariant Hom functor \( M \to \text{Hom}(M, X) \), with \( X = C \) and \( F : C \to X \) the identity, the exactness of the Hom sequence gives

\[
0 = F \circ g \circ f = g \circ f
\]

Thus, \( \text{Im} f \subseteq \ker g \). On the other hand, with \( X = B/\text{Im} f \) and \( F : B \to X \) the quotient map, by exactness of the Hom sequence there is \( F' : C \to X \) such that \( F' \circ g = F \). Thus, the kernel of \( g \) cannot be larger than \( \text{Im} f \), or \( F : B \to B/\text{Im} f \) could not factor through it. Thus, we have exactness. 

4. Half-exactness of adjoint functors

[4.0.1] Theorem: Let \( L, R \) be adjoint functors on \( \mathbb{Z} \)-modules, in the sense that there is a functorial isomorphism

\[
\text{Hom}(LA, B) \cong \text{Hom}(A, RB) \quad \text{(for every } A, B \text{)}
\]

Then \( L \) is right half-exact and \( R \) is left half-exact. That is, for

\[
0 \to A \to B \to C \to 0 \text{ exact } \implies LA \to LB \to LC \to 0 \text{ exact}
\]

and

\[
0 \to A \to B \to C \to 0 \text{ exact } \implies 0 \to RA \to RB \to RC \text{ exact}
\]

Proof: Left exactness of \( M \to \text{Hom}(X, M) \) for any \( X \) applies to \( X \) replaced by \( LX \), so

\[
0 \to \text{Hom}(LX, A) \to \text{Hom}(LX, B) \to \text{Hom}(LX, C) \text{ exact}
\]

By adjointness of \( L \) and \( R \), and functoriality of the adjointness isomorphisms, we have a commutative diagram with exact top row,

\[
\begin{array}{ccc}
0 & \to & \text{Hom}(LX, A) \\
\approx \downarrow & & \approx \downarrow \\
0 & \to & \text{Hom}(X, RA)
\end{array}
\]

Then the bottom row is exact, for all \( X \). By Yoneda’s lemma,

\[
0 \to RA \to RB \to RC \text{ exact}
\]

Similarly, for the other Hom functor, for all \( X \) we have a commutative diagram with exact top row,

\[
\begin{array}{ccc}
0 & \to & \text{Hom}(C, RX) \\
\approx \downarrow & & \approx \downarrow \\
0 & \to & \text{Hom}(LC, X)
\end{array}
\]

Then the bottom row is exact, for all \( X \), and by Yoneda

\[
LA \to LB \to LC \to 0 \text{ exact}
\]

since this second Hom functor \( M \to \text{Hom}(M, X) \) is contravariant.

[4.0.2] Corollary: The natural (adjointness) isomorphism \( \text{Hom}(A \otimes X, B) \approx \text{Hom}(A, \text{Hom}(X, B)) \) yields the left exactness of \( M \to \text{Hom}(X, M) \) and the right exactness of \( M \to M \otimes X \). 

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