Complex analysis examples discussion 02

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[02.1] Parametrize counter-clockwise a circle $\gamma$ of radius $r > 0$ centered at $z_o$, and directly compute $\int_\gamma (z-z_o)^n\,dz$ for all positive and negative integers $n$.

Such a path can be parametrized as $\gamma(t) = z_o + re^{it}$ for $0 \leq t \leq 2\pi$. Then

$$\int_\gamma (z-z_o)^n\,dz = \int_0^{2\pi} (re^{it})^n\,d(re^{it}) = \int_0^{2\pi} (re^{it})^n\,ire^{it}\,dt$$

$$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)t}\,dt = \begin{cases} [it]^{2\pi}_0 & = 2\pi i \text{ (for } n = -1) \\ \left[ire^{n+1}t\cdot e^{i(n+1)t}\right]^{2\pi}_0 & = 0 \text{ (for } n \neq -1) \end{cases}$$

[02.2] Using only geometric series expansions, determine the Laurent expansion of $f(z) = 1/(z - 1)(z - 2)$ in the annulus $1 < |z| < 2$, and also in the annulus $|z| > 2$.

By partial fractions, for $1 < |z| < 2$, expanding geometric series,

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{1-z} - \frac{1}{z}\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \ldots\right) - \frac{1}{z}\left(1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \ldots\right)$$

$$= -\frac{1}{2} - \sum_{n=1}^{\infty} \left(1 + \frac{1}{2n+1}\right) z^{-n} \quad \text{(in the annulus } 1 < |z| < 2)$$

For $|z| > 2$, the $1/(z-2)$ requires slightly different treatment:

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{2}\frac{1}{1-\frac{2}{z}} + \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \frac{1}{2}\left(1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \ldots\right) - \frac{1}{z}\left(1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \ldots\right)$$

$$= \sum_{n=1}^{\infty} (2^{n-1} - 1) z^{-n} = \sum_{n=2}^{\infty} (2^{n-1} - 1) z^{-n} \quad \text{(in the annulus } 1 < |z| < 2)$$

[02.3] Determine the Laurent expansion of $f(z) = 1/(z - 1)^4$ in the annulus $|z| > 1$, and also in the annulus $|z - 1| > 0$.

In $|z| > 1$,

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \frac{1}{z} \cdot \left(1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \ldots\right) = \frac{1}{z} + \frac{1}{z}^2 + \ldots$$

Differentiating termwise three times gives

$$\frac{(-1)(-2)(-3)}{(z-1)^4} = \frac{(-1)(-2)(-3)}{z^4} + \frac{(-2)(-3)(-4)}{z^5} + \ldots + \frac{(-n)(-n-1)(-n-2)}{z^{n+3}} + \ldots$$

which simplifies to

$$\frac{1}{(z-1)^4} = \frac{1}{z^4} + \frac{2\cdot3\cdot4/6}{z^5} + \ldots + \frac{n(n+1)(n+2)/6}{z^{n+3}} + \ldots$$
In the annulus $|z - 1| > 0$, the given expression $f(z) = (z - 1)^{-4}$ is already the Laurent expansion.

[02.4] Show that an entire function $f$ satisfying $|f(z)| \leq C \cdot (1 + |z|)^{1/2}$ for some constant $C$ is constant.

The argument is nearly identical to that of Liouville’s theorem that bounded entire functions are constant. Namely, Cauchy’s formula for the $n^{th}$ power series coefficient $c_n$ of $f$ at $0$, via a circle $γ_R$ of radius $R$ for any $R > 0$, is

$$c_n = \frac{1}{2\pi i} \int_{γ_R} \frac{f(w) \, dw}{w^{n+1}} \leq \text{length}(γ_R) \cdot \sup_{w \in γ_R} \frac{|f(w)|}{|w^{n+1}|} \leq \pi R \cdot \sup_{w \in γ_R} \frac{C \cdot (1 + R)^{1/2}}{R^{n+1}}$$

by the trivial estimate on the absolute value of a path integral. For $0 < n \in \mathbb{Z}$ this goes to 0 as $R \to +∞$, so all but the $0^{th}$ power series coefficient are 0. Since $f$ is entire, it is represented on the whole plane by its power series, so is constant.

[02.5] Compute $\int_{-∞}^{∞} \frac{dx}{x^4 + 1}$.

First, the infinite integral is a limit of finite limits

$$\int_{-∞}^{∞} \frac{dx}{x^4 + 1} = \lim_{R \to +∞} \int_{-R}^{R} \frac{dx}{x^4 + 1}$$

Note that the denominator has zeros at eighth roots of unity, namely, $ζ = e^{πi/4}$, $ζ^3 = e^{3πi/4}$, $ζ^5 = e^{5πi/4}$, $ζ^7 = e^{7πi/4}$. Let $γ_R$ be the path from $-R$ to $R$ along the real line, and then along the arc of the circle of radius $R$ in the upper half-plane, from $+R$ back to $-R$. The integral over the arc is estimated via the trivial estimate:

$$\left| \int_{arc \ R} \frac{dx}{x^4 + 1} \right| \leq \text{length} (arc \ R) \cdot \sup_{on \ arc \ R} \left| \frac{1}{z^4 + 1} \right| \leq \pi R \cdot \frac{1}{R - 1^4}$$

This goes to 0 as $R \to +∞$. Thus, using the Residue Theorem, the original integral is

$$\int_{-∞}^{∞} \frac{dx}{x^4 + 1} = \lim_{R \to +∞} \int_{γ_R} \frac{dz}{1 + z^4} = \lim_{R \to +∞} 2πi \text{Res}_{z=ζ,ζ^3,ζ^5,ζ^7} \frac{1}{1 + z^4}$$

recalling the convenient fact that the residue at $z_o$ of $g(z)/z - z_o$ for $g$ holomorphic at $z_o$ is $g(z_o)$. This is

$$2πi \left( \frac{1}{(v^2)(2ζ)(i\sqrt{2})} + \frac{1}{(-v^2)(i\sqrt{2})(2ζ)} \right) = \frac{πi}{2} \left( \frac{1}{iζ} + \frac{1}{ζ} \right) = \frac{πζ^2}{2} \cdot \frac{1 - i}{ζ} = \frac{π}{2} \cdot \frac{1 + i}{\sqrt{2}} \cdot (1 - i) = \frac{π}{\sqrt{2}}$$

[02.6] Compute $\int_{-∞}^{∞} \frac{e^{itx} \, dx}{x^4 + 1}$ with real $t$.

As in the previous example, the infinite integral is a limit of finite limits

$$\int_{-∞}^{∞} \frac{e^{itx} \, dx}{x^4 + 1} = \lim_{R \to +∞} \int_{-R}^{R} \frac{e^{itx} \, dx}{x^4 + 1}$$

The denominator has zeros at eighth roots of unity $ζ = e^{πi/4}$, $ζ^3$, $ζ^5$, $ζ^7$. Let $γ_R$ be the path from $-R$ to $R$ along the real line, and then along the arc of the circle of radius $R$ in the upper half-plane, from $+R$ back to $-R$. The integral over the arc is estimated via the trivial estimate:

$$\left| \int_{arc \ R} \frac{e^{itx} \, dx}{x^4 + 1} \right| \leq \text{length} (arc \ R) \cdot \sup_{on \ arc \ R} \left| \frac{e^{itx}}{z^4 + 1} \right| \leq πR \cdot \frac{e^{t \cdot -\text{Im}(z)}}{(R - 1)^4}$$
For $t \geq 0$, this goes to $0$ as $R \to +\infty$. Using the Residue Theorem, the original integral is

$$
\int_{-\infty}^{\infty} \frac{e^{itx}}{x^4+1} \, dx = \lim_{R \to +\infty} \int_{\gamma_R} \frac{e^{itz}}{1+z^4} = \lim_{R \to +\infty} 2\pi i \text{Res}_{z=\zeta} \frac{e^{itz}}{1+z^4}
$$

$$= 2\pi i \left( \frac{e^{it\zeta}}{(\zeta - \zeta^3)(\zeta - \zeta^5)(\zeta - \zeta^7)} + \frac{e^{it\zeta^3}}{(\zeta^3 - \zeta)(\zeta^3 - \zeta^5)(\zeta^3 - \zeta^7)} \right) \quad \text{(for } t \geq 0)$$

recalling the convenient fact that the residue at $z_o$ of $g(z)/(z-z_o)$ for $g$ holomorphic at $z_o$ is $g(z_o)$. The denominators simplify somewhat:

$$(\zeta - \zeta^3)(\zeta - \zeta^5)(\zeta - \zeta^7) = (\sqrt{2})(2\zeta)(i\sqrt{2}) = 4i\zeta
$$

and

$$(\zeta^3 - \zeta)(\zeta^3 - \zeta^5)(\zeta^3 - \zeta^7) = 4\zeta$$

so the $t \geq 0$ case gives

$$2\pi i \left( \frac{e^{it\zeta}}{4i\zeta} + \frac{e^{it\zeta^3}}{4\zeta} \right) = \frac{\pi}{2\zeta} e^{it\zeta} + \frac{\pi i}{2\zeta} e^{it\zeta^3} \quad \text{(for } t \geq 0)$$

For $t < 0$, replacing $x$ by $-x$ in the original integral reduces to the previous case. That is,

$$
\int_{-\infty}^{\infty} \frac{e^{itx}}{x^4+1} \, dx = \frac{\pi}{2\zeta} e^{it\zeta} + \frac{\pi i}{2\zeta} e^{it\zeta^3} \quad \text{(for } t \geq 0)
$$

[02.7] Compute $\int_0^\infty \frac{x \, dx}{1+x^3}$

As usual, the integral is the limit of finite integrals $\int_0^R$ as $R \to +\infty$. Let $\gamma_R$ be the path from $0$ to $R$ along the real line, then counter-clockwise along the circle of radius $R$ to $R \cdot e^{2\pi i/3}$, then back along the straight line to $0$. This path is chosen because the integral from $R \cdot e^{2\pi i/3}$ to $0$ is very simply related to the original:

$$
\int_R^0 \frac{e^{2\pi i/3}t}{1+(e^{2\pi i/3}t)^3} \, dt = -e^{4\pi i/3} \int_0^R \frac{t \, dt}{1+t^3}
$$

The integral along the arc is easily estimate by the trivial estimate:

$$
\left| \int_{\text{arc } R} \frac{z \, dz}{1+z^3} \right| \leq \text{length(arc } R) \cdot \sup_{\text{on arc } R} \left| \frac{z}{1+z^3} \right| \leq \frac{2\pi R}{3} \cdot \frac{R}{(R-1)^3}
$$

which goes to $0$ as $R \to +\infty$. The integral over $\gamma_R$ can be evaluated by residues: for $R > 1$, there is a single singularity inside $\gamma_R$, at the sixth root of unity $\zeta = \zeta_6 = e^{\pi i/3}$. Noting that

$$z^3 + 1 = (z+1)(z^2-z+1) = (z+1)(z-\zeta)(z-\zeta^{-1})$$

and that $-e^{4\pi i/3} = \zeta$, we have

$$(1 + \zeta) \int_0^\infty \frac{x \, dx}{1+x^3} = \lim_{R \to +\infty} \int_{\gamma_R} \frac{z \, dz}{1+z^3} = 2\pi i \text{Res}_{z=\zeta} = \frac{\zeta}{1+\zeta^3} = 2\pi i \frac{\zeta}{(\zeta+1)(\zeta-\zeta^{-1})}
$$

so
\[ \int_0^\infty \frac{x \, dx}{1 + x^3} = 2\pi i \frac{\zeta}{(\zeta + 1)^2(\zeta - \zeta^{-1})} = 2\pi i \frac{1}{(\zeta + 1)(\zeta^{-1} + 1)(i\sqrt{3})} = \frac{2\pi}{(\frac{1}{4} + \frac{i}{4})\sqrt{3}} = \frac{2\pi}{3\sqrt{3}} \]

[02.8] Compute \[ \frac{1}{1 + \frac{1}{24} + \frac{1}{36} + \frac{1}{44} + \ldots} \]

Arrange to evaluate the infinite sum by residues, by using the function \(2\pi i/(e^{2\pi iz} - 1)\), which we will check has simple poles with residues 1 at integers, and for \(z\) bounded away from integers is bounded. Granting that for a moment, letting \(\gamma_T\) be a counter-clockwise path around the square with vertices \(\pm T \pm iT\) with \(T \in \frac{1}{2} + \mathbb{Z}\), by residues

\[ \int_{\gamma_T} \frac{2\pi i}{e^{2\pi iz} - 1} \cdot \frac{1}{z^4} \, dz = \sum_{0 \leq |n| < T} \text{Res}_{z=n} \frac{2\pi i}{e^{2\pi iz} - 1} \cdot \frac{1}{z^4} = \sum_{0 < |n| < T} \frac{1}{n^4} + \text{Res}_{z=0} \frac{2\pi i}{e^{2\pi iz} - 1} \cdot \frac{1}{z^4} \]

Because of the division by \(z^4\), the latter residue is visibly the coefficient of \(z^3\) in the Laurent expansion of \(2\pi i/(e^{2\pi iz} - 1)\), which is determined by expanding \(1/(e^z - 1)\)

\[ \frac{1}{e^z - 1} = \frac{1}{1 + \left(\frac{z}{2} + \frac{z^2}{6} + \ldots\right)} = \frac{1}{1 - \left(\frac{z}{2} + \frac{z^2}{6} + \ldots\right)^2} = \frac{1}{1 - \left(\frac{z}{2} + \frac{z^2}{6} + \ldots\right)^3} + \ldots \]

The \(z^3\) coefficient of the latter is

\[ -\frac{1}{3!} + \left(\frac{2}{2!} \cdot \frac{1}{4!} + 1 \cdot \frac{1}{3!}\right)^2 - 3 \cdot \left(\frac{1}{2!}\right)^2 \cdot \frac{1}{3!} + \left(\frac{1}{2!}\right)^4 = -\frac{1}{120} + \frac{1}{24} + \frac{1}{36} - \frac{1}{8} + \frac{1}{16} \]

Replacing \(z\) by \(2\pi iz\) in that Laurent expansion, and multiplying from the \(2\pi i\) from the numerator multiplies this by \((2\pi i)^4 = 16\pi^4\), giving

\[ -\frac{16}{120} + \frac{16}{24} + \frac{16}{36} - \frac{16}{8} + \frac{16}{16} = -\frac{2}{15} + \frac{2}{3} + \frac{4}{9} - 1 = -\frac{6 + 30 + 20 - 45}{45} = -\frac{1}{45} \]

Thus, still granting that everything works out, we have

\[ \int_{\gamma_T} \frac{2\pi i}{e^{2\pi iz} - 1} \cdot \frac{1}{z^4} \, dz = \sum_{0 < |n| < T} \frac{1}{n^4} + \text{Res}_{z=0} \frac{2\pi i}{e^{2\pi iz} - 1} \cdot \frac{1}{z^4} = \sum_{0 < |n| < T} \frac{1}{n^4} - \frac{1}{45} \]

Taking the limit, the integral goes to 0, so

\[ 0 = \lim_{T \to \infty} \sum_{0 < |n| < T} \frac{1}{n^4} - \frac{1}{45} = 2 \cdot \sum_{n \geq 1} \frac{1}{n^4} - \frac{1}{45} \]

giving the claimed result. For the other details:

The function \(2\pi i/(e^{2\pi iz} - 1)\) has no singularities unless the denominator is 0, which occurs exactly at integers. It is \(\mathbb{Z}\)-periodic, so to check that its residue at 0 is 1: as above,
\[
\frac{2\pi i}{e^{2\pi iz} - 1} = \frac{2\pi i}{(1 + (2\pi iz) + \frac{(2\pi iz)^2}{2} + \ldots) - 1} = \frac{2\pi i}{2\pi iz + \frac{(2\pi iz)^2}{2} + \ldots} = \frac{1}{z + \frac{2\pi iz^2}{2} + \ldots} = \frac{1}{z} \cdot \frac{1}{1 + \left(\frac{2\pi iz}{2} + \ldots\right)} = \frac{1}{z} \left(1 - \left(\frac{2\pi iz}{2} + \ldots\right) + \ldots\right)
\]

To check that this function is bounded for \(z\) away from integers, first observe that \(|e^{2\pi iz}| \leq \frac{1}{e}\) for \(\text{Im}(z) \geq \frac{1}{2\pi}\), and \(|e^{2\pi iz}| \geq e\) for \(\text{Im}(z) \leq -\frac{1}{2\pi}\). In both cases, \(e^{2\pi iz} - 1\) is bounded away from zero, so \(2\pi i/(e^{2\pi iz} - 1)\) is bounded.

For \(|\text{Im}(z)| \leq \frac{1}{2\pi}\), again use periodicity, to reduce to the set where \(|\text{Im}(z)| \leq \frac{1}{2\pi}, 0 \leq \text{Re}(z) \leq 1,\) and \(|z - 0| \geq \frac{1}{2}\) and \(|z - 1| \geq 1\). This set is compact, and \(2\pi i/(e^{2\pi iz} - 1)\) is continuous on it, so is bounded. This completes the checking of the background details to make things work.