Complex analysis examples 04

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[This document is http://www.math.umn.edu/~garrett/m/complex/examples_2014-15/cx_discussion_04.pdf]

[04.1] Compute $\int_0^\infty \frac{x^s}{1 + x^2} \, dx$

The integral is absolutely convergent for $-1 < \text{Re}(s) < 1$. Implicitly,

$$x^s = e^{s \log x}$$

where the logarithm is the one which is real-valued on $(0, +\infty)$. Use the Hankel/keyhole contour. First, the integral itself is a limit

$$\int_0^\infty \frac{x^s}{1 + x^2} \, dx = \lim_{\varepsilon \to 0^+, \, R \to +\infty} \int_{\varepsilon}^{R} \frac{x^s}{1 + x^2} \, dx$$

Let $H_{\varepsilon, R}$ be the Hankel/keyhole contour that comes from $R$ along the real line to $\varepsilon$, then traces a circle of radius $\varepsilon$ around 0 counter-clockwise to $\varepsilon$, then back out to $R$. Let $H_{\varepsilon}$ be the limiting case as $R \to +\infty$. We want the integral along that part of the path, the outbound part from $\varepsilon$ back out to $R$, to be the original integral $\int_{\varepsilon}^{R} \frac{x^s}{(x^2 + 1)} \, dx$. That is, we want the value of $x^s$ to match.

On that small circle, the argument of $x$ changes continuously, with a net increase of $2\pi$ from its value on the in-bound part of the path. Requiring that $x^s$ change continuously on that small circle, and be $e^{s \log x}$ with real-valued log $x$ after traversing $2\pi$ radians counter-clockwise, requires that $x^s$ be $e^{s(\log x - 2\pi i)}$ on the in-bound path. Thus,

$$\int_{\text{outbound+inbound}} \frac{x^2 \, dx}{1 + x^2} = (1 - e^{-2\pi i s}) \int_{\varepsilon}^{R} \frac{x^2 \, dx}{1 + x^2}$$

Further, the main point of the keyhole trick is that, surprisingly, the limit over $\varepsilon \to 0^+$ is reached in finite time, in the sense that there is sufficiently small $\varepsilon_0 > 0$ such that

$$\lim_{\varepsilon \to 0^+} \int_{H_{\varepsilon, R}} \frac{x^s \, dx}{1 + x^2} = \int_{H_{\varepsilon_1, R}} \frac{x^s \, dx}{1 + x^2} \quad \text{(for all positive } \varepsilon_1 < \varepsilon_0)$$

Recall the proof: for $0 < \varepsilon_1 < \varepsilon_0$, let $\gamma_{\varepsilon_0, \varepsilon_1}$ be the closed path that traces counter-clockwise around the circle of radius $\varepsilon_0$ from $\varepsilon_0$ back to $\varepsilon_0$, then left to $\varepsilon_1$, then clockwise around a circle of radius $\varepsilon_1$ back to $\varepsilon_1$, then right to $\varepsilon_0$. In the interior of this path, the integrand is holomorphic. Adding the integral over $\gamma_{\varepsilon_0, \varepsilon_1}$ to the integral over $H_{\varepsilon_1, R}$ makes the integrals from $\varepsilon_0$ to $\varepsilon_1$ (inbound) and from $\varepsilon_1$ to $\varepsilon_0$ (outbound) cancel, and the integrals around the circles of radius $\varepsilon_1$ cancel, leaving $H_{\varepsilon_0, R}$. (Yes, one should draw a picture.)

To evaluate

$$\int_{H_{\varepsilon_1, R}} \frac{x^s \, dx}{1 + x^2}$$

add an integral counter-clockwise around a circle $\sigma_R$ of radius $R$ from $R \in \mathbb{R}$ back to $R$. For $\text{Re}(s) < 1$, the trivial estimate on this integral is

$$\left| \int_{\sigma_R} \frac{x^s \, dx}{1 + x^2} \right| \leq \text{length} \cdot \sup_{\sigma_R} \left| \frac{x^s \, dx}{1 + x^2} \right| \leq 2\pi R \cdot \frac{R^{\text{Re}(s)}}{(R - 1)^2} \to 0 \quad \text{(as } R \to +\infty, \text{ for } \text{Re}(s) < 1)$$

Thus,

$$\lim_{R} \int_{H_{\varepsilon_1, R} + \sigma_R} \frac{x^s \, dx}{1 + x^2} = \int_{H_{\varepsilon_1} + \sigma_R} \frac{x^s \, dx}{1 + x^2} = (1 - e^{-2\pi i s}) \int_0^\infty \frac{x^s \, dx}{1 + x^2}$$

\(\square\)
On the other hand, the integral over the closed contour $H_{\varepsilon_1,R} + \sigma_R$ can be evaluated by residues: it is $-2\pi i$ times the sum of residues in its interior, since the boundary is traced clockwise. Inside that path, for small $\varepsilon_1$ and large $R$, there are exactly two poles, at $x = \pm i$, and both are simple. The value of $\arg x$ at $-i$ is obtained by moving clockwise from the arg $x = 0$ on $(0, +\infty)$, giving $-\frac{\pi}{2}$. The argument at $+i$ is obtained by continuing clockwise, giving $-\frac{3\pi}{2}$. Thus,

$$\text{sum of residues} = \frac{e^{-\frac{3\pi i}{2}}}{-i} + \frac{e^{-\frac{3\pi i}{2}}}{i} = \frac{e^{-\frac{3\pi i}{2}}}{-2i} + \frac{e^{-\frac{3\pi i}{2}}}{2i}$$

In summary,

$$\int_0^\infty \frac{x^s}{1 + x^2} \, dx = \frac{1}{1 - e^{-2\pi i s}} \lim_{R \to \infty} \int_{H_{\varepsilon_1} + \sigma_R} \frac{x^s}{1 + x^2} \, dx = \frac{-2\pi i}{1 - e^{-2\pi i s}} \left( e^{-\frac{\pi i}{2}} - e^{-\frac{3\pi i}{2}} \right)$$

$$= \frac{\pi}{1 - e^{-2\pi i s}} \left( e^{-\frac{\pi i s}{2}} - e^{-\frac{3\pi i s}{2}} \right) = \frac{\pi}{2} e^{\frac{\pi i s}{2}} - e^{\frac{3\pi i s}{2}} = \frac{\pi}{2} \cos \frac{\pi s}{2}$$

[04.2] Compute $\int_0^1 \frac{x(x(1-x))^s}{1 + x^3} \, dx$

Oops, as it stands, I don’t think that we can do much with it. Possibly what I intended, or in any case is better, was something like

$$\int_0^1 \frac{x^s (1-x)^{-s}}{1 + x^3} \, dx \quad (\text{with } \Re(s) > -1)$$

This does admit a variation of the Hankel/keyhole contour idea, namely, tracking $s \arg x$ counter-clockwise around $0$ adds $2\pi s$, while tracking $-s \arg(1-x)$ counter-clockwise around $1$ subtracts $2\pi s$. That is, moving around both $0, 1$ (with the modified set-up) returns $x^s (1-x)^{-s}$ to its original value. That is, on $\mathbb{C} - [0,1]$, the complex plane with the unit interval removed, there is a well-defined holomorphic (and genuinely single-valued!) $x^s (1-x)^{-s}$.

The original integral from $0$ to $1$ is not cancelled by the integral from $1$ to $0$ after going around $1$ counter-clockwise, because $x^s (1-x)^{-s}$ has become $e^{-2\pi is} \cdot x^s (1-x)^{-s}$. Thus, for $\varepsilon > 0$, letting $\gamma_\varepsilon$ be the path from $\varepsilon$ to $1 - \varepsilon$, then clockwise around $1$ back to $1 - \varepsilon$, then left to $\varepsilon$, and around $0$ counter-clockwise back to $\varepsilon$,

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\gamma_\varepsilon} \frac{x^s (1-x)^{-s}}{1 + x^3} \, dx = (1 - e^{-2\pi i s}) \int_0^1 \frac{x^s (1-x)^{-s}}{1 + x^3} \, dx$$

Let $\sigma_R$ be a circle of radius $R$, traversed clockwise. Connect $\sigma_R$ and $\gamma_\varepsilon$ by suitably oriented inbound and outbound paths to create a large path $\tau$. As usual, the inbound and outbound integrals are mutually cancelling. In the interior of $\tau$ the integrand is meromorphic, with simple poles at $-1$ and primitive sixth roots of $1$, $\zeta = e^{\pi i/3}$, and $\zeta^{-1} = e^{-\pi i/3}$. Thus, noting that the large path is negatively oriented, so that $-2\pi i$ times the residues is picked up,

$$\int_{\gamma_\varepsilon + \sigma_R} \frac{x^s (1-x)^{-s}}{1 + x^3} \, dx = -2\pi i \operatorname{Res}_{x=1, \zeta, \zeta^{-1}} \frac{x^s (1-x)^{-s}}{1 + x^3}$$

$$= -2\pi i \left( \frac{(-1)^s (1-(-1))^{-s}}{(1-\zeta)(-1-\zeta^{-1})} + \frac{\zeta^s (1-\zeta)^{-s}}{(1-\zeta)(\zeta^{-1}-1)} + \frac{(\zeta^{-1})^s (1-\zeta^{-1})^{-s}}{(1-\zeta^{-1})(\zeta-1)} \right)$$

Then there is the task of identifying the correct $s^{th}$ powers. Putting that off, the integral over $\sigma_R$ can be easily estimated for $\Re(s) < 0$ by

$$\left| \int_{\sigma_R} \frac{x^s (1-x)^{-s}}{1 + x^3} \, dx \right| \leq \text{length} \cdot \sup \text{on path} \leq 2\pi R \cdot \frac{(R+1)^2 \Re(s)}{(R-1)^3} \to 0 \quad (\text{as } R \to \infty)$$
Thus, with suitable values of \( s \)th powers,

\[
\int_0^1 \frac{x^s(1 - x)^{-s}}{1 + x^3} \, dx = \frac{-2\pi i}{1 - e^{-2\pi is}} \left( \frac{(1 - 1/\zeta)}{(-1 - \zeta)(1 - \zeta')} + \frac{(\zeta^{-1})}{(\zeta - (1))(\zeta - \zeta^-1)} + \frac{(\zeta'^{-1})}{(1 - \zeta^-1)(\zeta'^{-1} - \zeta^-1)} \right)
\]

Last, tracking args. Since \((x/(1 - x))^s\) is well-defined on \( \mathbb{C} - [0,1] \), it shouldn’t make any difference how we do this, as long as we consider \( x/(1 - x) \) as a single entity. Going from \([0,1]\) clockwise around 0 to \(-1\) decreases the argument of \( \frac{x}{1-x} \) from 0 to \(-\pi\). Thus,

\[
\left( \frac{-1}{1 - (-1)} \right)^s = \left( -\frac{1}{2} \right)^s = e^{s(-\log 2 - \pi i)} = 2^{-s}e^{\pi is}
\]

From \([0,1]\) clockwise to \( \zeta \) decreases the argument of \( \frac{x}{1-x} \) from 0 to

\[
\arg \left( \frac{\zeta}{1-\zeta} \right) = \arg \left( \frac{\zeta}{1-\zeta} \right) = \arg \zeta^2 = -\frac{4}{3}\pi
\]

Thus,

\[
\left( \frac{\zeta^{-1}}{1-\zeta^{-1}} \right)^s = (\zeta^{-2})^s = e^{s(-\frac{4}{3}\pi)}
\]

From \([0,1]\) clockwise to \( \zeta^{-1} \) decreases the argument of \( \frac{x}{1-x} \) from 0 to

\[
\arg \left( \frac{\zeta^{-1}}{1-\zeta^{-1}} \right) = \arg \left( \frac{\zeta^{-1}}{1-\zeta^{-1}} \right) = \arg \zeta^{-2} = -\frac{2}{3}\pi
\]

Thus,

\[
\int_0^1 \frac{x^s(1 - x)^{-s}}{1 + x^3} \, dx = \frac{-2\pi i}{1 - e^{-2\pi is}} \left( \frac{2^{-s}e^{\pi is}}{(1 + \zeta)(1 + \zeta')}) + \frac{e^{-\frac{4}{3}\pi is}}{(\zeta + 1)(i\sqrt{3})} + \frac{e^{-\frac{2}{3}\pi is}}{(\zeta^{-1} + 1)(-i\sqrt{3})} \right)
\]

Perhaps further simplification is of less interest... although one might hope to certify that for \( s \in \mathbb{R} \) this apparent outcome is real. \hspace{1cm} ///

\[04.3\] Compute \( \int_0^\infty e^{-i\xi x} x^se^{-x} \, dx \) with \( \text{Re}(s) > -1 \).

This invites application of the Gamma identity

\[
\int_0^\infty e^{-xy} x^s \, dx = y^{-s} \int_0^\infty e^{-x} x^s \, dx = y^{-s} \Gamma(s)
\]

which holds first for \( y > 0 \) and then for \( \text{Re}(y) > 0 \), by the Identity Principle (also known as The Permanence of Analytic Relationships):

\[
\int_0^\infty e^{-i\xi x} x^{s+1} e^{-x} \, dx = \int_0^\infty e^{-x(1+i\xi)} x^{s+1} e^{-x} \, dx = (1 + i\xi)^{-(s+1)} \cdot \Gamma(s + 1)
\]

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Thus,
\[
\int_{-\infty}^{\infty} e^{-i\xi x} \, dx = \int_{-\infty}^{\infty} e^{-i\xi x} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d}{dx} e^{-i\xi x} \cdot e^{-x^2} \, dx = -\frac{1}{2}i\xi \int_{-\infty}^{\infty} e^{-i\xi x} \, e^{-x^2} \, dx
\]

The exponentials can be combined, and then complete the square:
\[
\int_{-\infty}^{\infty} e^{-i\xi x} \, e^{-x^2} \, dx = \int_{-\infty}^{\infty} e^{-(x^2 + i\xi x - \frac{\xi^2}{4})} \, dx = e^{-\frac{\xi^2}{4}} \int_{-\infty}^{\infty} e^{-(x + \frac{i\xi}{2})^2} \, dx
\]

The intuition at this point is that sliding the integral from \(-\infty\) to \(+\infty\) along the real axis to be an integral from \(-i\xi - \infty\) to \(-i\xi + \infty\) will not change the value of the integral, since there are no residues to pick up, while it will convert the integrand back to \(e^{-x^2}\), which does not involve \(\xi\).

As usual, an integral from \(-\infty\) to \(+\infty\) is a limit of the corresponding integral from \(-R\) to \(+R\), as \(R \to +\infty\). Then
\[
\int_{-\infty}^{\infty} e^{-(x+\frac{i\xi}{2})^2} \, dx = \lim_{R \to \infty} \int_{-R}^{R} e^{-(x+i\xi)^2} \, dx = \int_{-\infty}^{\infty} e^{-(x+i\xi)^2} \, dx = e^{-\frac{\xi^2}{4}} \int_{-\infty}^{\infty} e^{-x^2} \, dx
\]

Let \(B_R\) be the rectangle with vertices \(\pm R\) and \(-i\xi \pm R\), traced counter-clockwise. The integrals over the ends of the box are easily estimated: since \(|e^{-(x+i\xi)^2}| = e^{-\Re((x+i\xi)^2)} = e^{-x^2+y^2}\),
\[
\left| \int_{R}^{\infty} e^{-x^2} \, dx \right| \leq \text{length} \cdot (\text{sup on curve}) \leq |\xi| \cdot e^{-R^2} \cdot e^{\xi^2} \to 0 \quad \text{as } R \to +\infty
\]

Thus,
\[
0 = \lim_{R \to \infty} 0 = \lim_{R \to \infty} \int_{B_R} e^{-i\xi x} \, e^{-x^2} \, dx = \lim_{R \to \infty} \left( e^{-\frac{\xi^2}{4}} \int_{-R}^{R} e^{-x^2} \, dx \right) = e^{-\frac{\xi^2}{4}} \cdot \int_{-\infty}^{\infty} e^{-x^2} \, dx
\]

and
\[
\int_{-\infty}^{\infty} e^{-i\xi x} \, e^{-x^2} \, dx = -\frac{1}{2}i\xi \int_{-\infty}^{\infty} e^{-i\xi x} \, e^{-x^2} \, dx = -\frac{1}{2}i\xi \cdot e^{-\frac{\xi^2}{4}} \cdot \sqrt{\pi}
\]

[04.5] For continuous \(\varphi\) on the unit circle \(|z| = 1\), define
\[
f_\varphi(z) = \int_{0}^{2\pi} \frac{\varphi(e^{i\theta})}{e^{i\theta} - z} \, d\theta \quad \text{ (for } |z| < 1\text{)}
\]

Show that \(f(z)\) is holomorphic. Give an example of \(\varphi\) not identically \(0\) so that \(f_\varphi\) is identically \(0\).

Use Morera’s theorem: with \(\gamma\) be a small counter-clockwise triangle around a given \(z_o\) in the open unit disk,
\[
\int_{\gamma} f_\varphi(z) \, dz = \int_{\gamma} \int_{0}^{2\pi} \frac{\varphi(e^{i\theta})}{e^{i\theta} - z} \, d\theta \, dz = \int_{0}^{2\pi} \varphi(e^{i\theta}) \left( \int_{\gamma} \frac{dz}{e^{i\theta} - z} \right) \, d\theta = \int_{0}^{2\pi} 0 \, d\theta = 0
\]

Morera’s theorem says that this vanishing implies holomorphy of \(f_\varphi\).
Note that the given integral is not quite a written-out version of Cauchy’s kernel, because \( d(e^{i\theta}) = i\theta \, e^{i\theta} \, d\theta \), so a factor of \( e^{i\theta} \) is missing. Nevertheless, it’s close. Thus, various heuristics might suggest making \( \varphi(e^{i\theta}) \) be the boundary value of an anti-holomorphic function such as \( F(z) = \pi \). Thus, \( \varphi(e^{i\theta}) = F(e^{i\theta}) - e^{-i\theta} \). For \(|z| < 1\), expanding a geometric series:

\[
\varphi(z) = \int_0^{2\pi} \frac{\varphi(e^{i\theta})}{e^{i\theta} - z} \, d\theta = \int_0^{2\pi} \frac{e^{-i\theta}}{e^{i\theta} - z} \, d\theta = \int_0^{2\pi} \frac{e^{-i\theta}}{1 - ze^{-i\theta}} \, d\theta = \sum_{n=0}^{\infty} \int_0^{2\pi} e^{-2i\theta} (ze^{-i\theta})^n \, d\theta \\
= \sum_{n=0}^{\infty} z^n \int_0^{2\pi} e^{-i(2+n)\theta} \, d\theta = \sum_{n=0}^{\infty} z^n \cdot 0 = 0
\]

Thus, with hindsight, \( \varphi(e^{i\theta}) = 1 \) would also have given \( \varphi = 0 \).

\[04.6\] Let \( f \) be an entire function such that \( f(z + 1) = f(z) \) and \( f(z + i) = f(z) \) for all \( z \). Show that \( f \) is constant.

First, the given periodicity relations imply that all the values of \( f \) are determined by its values on \( R = \{z = x + iy : 0 \leq x \leq 1, \, 0 \leq y \leq 1\} \): given \( x, y \), there are unique integers \( m, n \) such that \( m \leq x < m + 1 \) and \( n \leq y < n + 1 \). By the obvious induction,

\[
f(x + iy) = f((x - m) + i(y - n))
\]

while \( 0 \leq x - m < 1 \) and \( 0 \leq y - n < 1 \). On the compact set \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 \), the continuous function \( f \) is bounded. Thus, \( f \) is entire and bounded, so by Liouville, it is constant.

\[04.7\] Show that a real-valued holomorphic function is constant.

For \( f \) real-valued on a neighborhood of \( z_0 \), taking a derivative along a real direction, but also along a purely imaginary direction, gives

\[
f'(z_0) = \lim_{\varepsilon \to 0} \frac{f(z_0 + \varepsilon) - f(z_0)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{f(z_0 + i\varepsilon) - f(z_0)}{i\varepsilon} \quad \text{(with \( \varepsilon \) real)}
\]

The first limit is real, the second imaginary, so the equality implies that they are both 0. Thus, \( f' = 0 \), and \( f \) is constant.

Another kind of argument, applicable to entire functions with constrained values: for \( f \) were entire and real-valued, the function \( F(z) = e^{if(z)} \) takes values on the unit circle. In particular, \( F \) is bounded and entire, so constant, by Liouville. Then \( 0 = F'(z) = if'(z)e^{if(z)} \), so \( f'(z) = 0 \), and \( f \) is constant.

\[04.8\] The Bergmann kernel of the unit disk is

\[
K(z, w) = \frac{1}{\pi} \frac{1}{(1 - wz)^2}
\]

For \( f \) holomorphic on the open unit disk and extending continuously to a continuous function on the closed unit disk, show that

\[
f(w) = \int \int_{x^2 + y^2 \leq 1} f(x + iy) \, K(z, w) \, dx \, dy
\]

In fact, it is better to derive the kernel from first principles. That is, holomorphic functions on the unit disk that extend to be continuous on the closed disk are bounded, so we can put the hermitian inner product

\[
\langle f, g \rangle = \int \int_{x^2 + y^2 \leq 1} f(x + iy) \, \overline{g(x + iy)} \, dx \, dy
\]
on the \( \mathbb{C} \)-vectorspace of such functions. It is natural to wonder about \( \langle z^m, z^n \rangle \):

\[
\langle z^m, z^n \rangle = \iint_{x^2 + y^2 \leq 1} z^m \overline{z}^n \, dx \, dy = \int_0^1 \int_0^{2\pi} r^{m+n} e^{\pi i (n-m)\theta} \, d\theta \, r \, dr
\]

\[
= \delta_{mn} 2\pi \int_0^1 r^{2n} \, dr = \frac{\pi}{n+1} \delta_{mn}
\]

with \( \delta_{mn} = 1 \) if \( m = n \) and 0 otherwise. Thus, \( u_n(z) = z^n \cdot \sqrt{\frac{n+1}{\pi}} \) is an orthonormal basis, and the reproducing kernel, or Bergmann kernel, is

\[
K(z, w) = \sum_n u_n(z) \cdot u_n(w) = \frac{1}{\pi} \sum_{n=0}^{\infty} (n+1) z^n \overline{w}^n = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{1}{n} \frac{d}{dz} (z \overline{w})^{n+1}
\]

\[
= \frac{1}{\pi} \frac{1}{\bar{w}} \frac{d}{dz} \frac{z}{1-z \overline{w}} = \frac{1}{\pi} \frac{1}{(1-z \overline{w})^2}
\]

Just to check, use the power series expansion \( f(z) = \sum_{n \geq 0} c_n z^n \), expand the kernel as a geometric series

\[
\frac{1}{\pi} \frac{1}{(1-wz)^2} = \frac{1}{\pi} \frac{1}{w} \frac{d}{dz} \frac{1}{1-wz} = \frac{1}{\pi} \frac{1}{w} \frac{d}{dz} \left( 1 + wz + (wz)^2 + \ldots \right) = \frac{1}{\pi} \left( 1 + 2\pi z + 3(\overline{w} z)^2 + \ldots \right)
\]

Then the integral is

\[
\frac{1}{\pi} \sum_{m \geq 0, n \geq 0} c_n \int_0^1 \int_0^{2\pi} r^{m+n} (m+1)w^m e^{\pi i (n-m)\theta} \, d\theta \, r \, dr = \frac{1}{\pi} \sum_{n \geq 0} c_n 2\pi (n+1)w^n \int_0^1 r^{2n} \, dr
\]

\[
= \sum_{n \geq 0} c_n 2(n+1)w^n \frac{1}{2n+2} = \sum_{n \geq 0} c_n w^n = f(w)
\]