(*November 23, 2014*)

Complex analysis discussion 06

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[This document is http://www.math.umn.edu/˜garrett/m/complex/examples_2014-15/cx_discussion_06.pdf]

[06.1] Evaluate $\int_0^{2\pi} \frac{dt}{2+\cos t}$

Convert this to a contour integral over the path γ traversing the unit circle counter-clockwise:

$$\int_0^{2\pi} \frac{dt}{2+\cos t} \;=\; \int_0^{2\pi} \frac{ie^{it}\,dt}{ie^{it}\cdot(2+\frac{e^{it}+e^{-it}}{2})} \;=\; -2i\int_\gamma \frac{dz}{4z+z^2+1} \;=\; (-2i)\cdot(2\pi i)\cdot(\text{sum of residues inside }\gamma)$$

Solving $z^2+4z+1=0$, $z=\frac{-4\pm\sqrt{4^2-4}}{2}=-2\pm\sqrt{3}$. Of these two values, only $-2+\sqrt{3}$ is inside the unit circle, so the integral is

$$4\pi^2\cdot\frac{1}{(-2+\sqrt{3})-(-2-\sqrt{3})} \;=\; \frac{2\pi^2}{\sqrt{3}}$$

[06.2] Show that $z^{10}-z^7+4z^2-1=0$ has exactly two zeros inside the circle $|z|=1$.

This invites application of Rouché's theorem, and, indeed,

$$\Big|4z^2-(z^{10}-z^7+4z^2-1)\Big| \;=\; \Big|-z^{10}+z^7+1\Big| \;\le\; 3 \;<\; 4 \;=\; |4z^2| \qquad (\text{on } |z|=1)$$

Since $4z^2$ has exactly two zeros inside $|z|=1$, by Rouché, the same is true of $z^{10}-z^7+4z^2-1$. ///

[06.3] Show that $\cos z$ has exactly two complex zeros inside $|z|=2$ by comparing it to $1-\frac{z^2}{2}$, which certainly has exactly two complex zeros inside that circle.

We suspect that this will follow from Rouché. Indeed,

$$\Big|(1-\frac{z^2}{2})-\cos z\Big| \;=\; \Big|\frac{z^4}{4!}-\frac{z^6}{6!}+\ldots\Big| \;\le\; \frac{16}{24}\cdot(1+\frac{4}{5\cdot 6}+\frac{16}{5\cdot6\cdot7\cdot8}+\ldots)$$

$$\le\; \frac{16}{24}\cdot(1+\frac{2}{15}+(\frac{2}{15})^2+\ldots) \;=\; \frac{16}{24}\cdot\frac{1}{1-\frac{2}{15}} \;=\; \frac{2}{3}\cdot\frac{15}{13} \;=\; \frac{10}{13} \;<\; 1 \;=\; 2-1 \;\le\; |1-\frac{z^2}{2}|$$

so $\cos z$ has exactly two compact zeros inside $|z|=2$, since $1-\frac{z^2}{2}$ does.

[06.4] Prove that, given holomorphic f,g on a non-empty open set U, and given a *simple* zero z_o of f, for all small-enough complex ε the zero of $f+\varepsilon g$ nearest z_o is also *simple*.

(This is a slightly more precise version of a result proven earlier.) Let $|z-z_o|=r>0$ be a small circle so that it and its interior are entirely within U. Shrink $r>0$ if necessary so that f does not vanish on the circle, by the identity principle. Let $m>0$ be the minimum of the continuous function $|f|$ on the circle, and M the maximum of the continuous function $|g|$ on the circle. Then, on that circle, for $0<\varepsilon<\frac{m}{M}$,

$$\Big|f-(f+\varepsilon\cdot g)\Big| \;\le\; \varepsilon\cdot|g| \;<\; \frac{m}{M}\cdot M \;=\; m$$

Thus, Rouché's theorem says that the number of zeros of $f+\varepsilon g$ inside the circle is equal to the number of zeros of f there, namely, one. ///

[06.5] Let U be the region

$$U = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0,\ |z-(1+i)| > 1,\ |z-(1-i)| > 1\}$$

Let $\widetilde{U}$ be the topological closure of U with the point 1 removed. (Thus, $\widetilde{U}$ includes the interval $[-i,+i]$ along the imaginary axis, and two quarter-circles with the point 1 removed.) Construct a holomorphic function on U extending to a continuous function on $\widetilde{U}$, bounded by 1 on the boundary *except for* 1, but *unbounded* on U.

This region is a slightly disguised version of a holomorphic image of a half-strip, to which we could apply information from the context of the Phragmén-Lindelöf theorem: $z \to \frac{z+1}{z-1}$ maps $1 \to \infty$, $i \to \frac{i+1}{i-1} = i$, $-i \to \frac{-i+1}{-i-1} = -i$, and $-1 \to 0$. *Anticipating* that linear fractional transformations map lines-and-circles to lines-and-circles, and that these are determined by three distinct points on them, this map sends U to a subregion of the strip $\mathrm{Re}(z) > 0$ and $-1 < \mathrm{Im}(z) < 1$.

The key example-idea $z \to e^{e^{\pi z}}$ is bounded on the upper and lower boundaries of that strip:

$$\left|e^{e^{\pi(x\pm i)}}\right| = e^{\mathrm{Re}\,(e^{\pi(x\pm i)})} = e^{-e^{\pi x}} \leq 1$$

However, as planned, on $\mathrm{Im}(z) = 0$ the function blows up as $\mathrm{Re}(z) \to +\infty$. ///

[06.6] Let C be the usual Cantor set

$$C = \{x \in [0,1] : \text{the ternary expansion of } x \text{ contains only digits 0 and 2, digit 1}\}$$

where terminal repeating 1's $(\ldots 111111\ldots)$ are converted to $\ldots 2$. Show that there is no non-constant holomorphic function with real part taking values in C.

One decisive approach is to invoke the open mapping theorem: images of opens under non-constant holomorphic functions are open. The Cantor set contains no non-empty open subsets. ///

[06.7] Given $R > 0$, $w_o \in \mathbb{C}$, and $\varepsilon > 0$, show that there is $z \in \mathbb{C}$ with $|z| > R$ and $|e^z - w_o| < \varepsilon$.

This is an application of the Casorati-Weierstraßtheorem, namely, that near an essential singularity a holomorphic function comes arbitrarily close to every complex number. At ∞, $z \to e^z$ has the same behavior as $z \to e^{1/z}$ at 0. The power series expansion of $z \to e^z$ gives the Laurent expansion of $z \to e^{1/z}$ at 0, and it has infinitely-many terms. Thus, the singularity there is *essential*. ///

[06.8] For small $w \in \mathbb{C}$, let $f(w)$ be the simple zero of $z^5 - z + w = 0$ near 0. Determine a few terms of the power series expansion of $f(w)$ at $w = 0$.

There are at least two superficially different ways to approach this. In both cases, we presume that the simple zero really is a holomorphic function of w. Indeed, since $w = -z^5 + z$ is a holomorphic function of z, and $(-z^5+z)' = -5z^4 + 1$ is $1 \neq 0$ at $z = 0$, by the holomorphic inverse function theorem, the simple zero really is a holomorphic function of w near 0.

One way is to let $f(w) = c_1w + c_2w^2 + \ldots$, substitute $f(w)$ for z in the equation, and get a recursive expression for c_n. The other is to implicitly differentiate. Perhaps we should try both, to compare the workloads.

Substituting the power series for z,

$$\Big(c_1w + c_2w^2 + c_3w^3 + \ldots\Big)^5 - \Big(c_1w + c_2w^2 + c_3w^3 + \ldots\Big) + w = 0$$

The lowest-order term of that fifth power only appears at degree 5, so $c_1 = 1$, $c_2 = c_3 = c_4 = 0$, by *uniqueness* of power-series expressions for holomorphic functions. Then the implied relation from the equality of w^5 terms is $c_1^5 - c_5 = 0$, so $c_5 = c_1^5 = 1$.

Subsequently, the first place that c_n appears is in the equality for coefficients of w^n, which in the $f(w)^5$ term can only at highest the coefficient c_{n-4}. Thus, there will be a recursion that produces all subsequent coefficients. Note that existence and uniqueness were not in doubt after invocation of the holomorphic inverse function theorem. ///

The other approach, by implicit differentiation, computing power series coefficients by computing successive derivatives, first notes that $f(0) = 0$ satisfies $f(0)^5 - f(0) + 0 = 0$, so we do take $c_0 = 0$. Differentiating with respect to w,

$$0 \;=\; 5f'(w) \cdot f(w)^4 - f'(w) + 1$$

so

$$f'(w) \;=\; \frac{1}{1 - 5 \cdot f(w)^4}$$

and

$$f'(0) \;=\; \frac{1}{1 - 5 \cdot 0} \;=\; 1$$

Differentiating again,

$$f''(w) \;=\; \frac{-5 \cdot 4 \cdot f'(w) \cdot f(w)^3}{(1 - 5 \cdot f(w)^4)^2}$$

At $w = 0$, with $f(0) = 0$, this is 0. And so on. Perhaps this approach makes less clear that several power series coefficients are clearly determined (and are 0), than the substitute-and-solve approach, although that is partly due to the sparseness of the polynomial. ///
