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Complex analysis discussion 07

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[This document is http://www.math.umn.edu/˜garrett/m/complex/examples_2014-15/cx_discussion_07.pdf]

[07.1] Exhibit a linear fractional transformation mapping $1, 2, 3$ to z_1, z_2, z_3.

Presumably the z_i are distinct, or else this is impossible. We know the qualitative fact that linear fractional transformations are transitive on triples of distinct points on $\mathbb{CP}^1$, and this question is asking for a formula, which will involve variants of what was classically called the *cross ratio*.

An unglamorous but systematic approach is to map one triple of distinct numbers to $0, 1, \infty$, and then back from $0, 1, \infty$ to the other, or similar. There are various computational approaches to obtaining $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ mapping given z_1, z_2, z_3 to $0, 1, \infty$. One approach is to first map $z_3 \to \infty$ and $z_1 \to 0$, which is easily done via $\begin{pmatrix} 1 & -z_1 \\ 1 & -z_3 \end{pmatrix}$. This sends $z_2 \to \dfrac{z_2 - z_1}{z_2 - z_3}$. To subsequently send the latter to 1 while stabilizing 0 and ∞, multiply by the multiplicative inverse of the latter complex number. Thus,

$$z \longrightarrow \frac{z - z_1}{z - z_3} \longrightarrow \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1} = \frac{z_2 - z_3}{z_2 - z_1} \begin{pmatrix} 1 & -z_1 \\ 1 & -z_3 \end{pmatrix} (z) \qquad \text{(sends } z_1, z_2, z_3 \text{ to } 0, 1, \infty)$$

The matrix inverse is

$$\begin{pmatrix} 1 & -z_1 \\ 1 & -z_3 \end{pmatrix}^{-1} = \frac{1}{-z_3 + z_1} \begin{pmatrix} -z_3 & z_1 \\ -1 & 1 \end{pmatrix} (z)$$

Thus,

$$z \longrightarrow \begin{pmatrix} -z_3 & z_1 \\ -1 & 1 \end{pmatrix} \Big(\frac{z_2 - z_1}{z_2 - z_3} \cdot z \Big) \qquad \text{(sends } 0, 1, \infty \text{ to } z_1, z_2, z_3)$$

We can send $1, 2, 3$ to $0, 1, \infty$ by

$$z \longrightarrow \frac{z - 1}{z - 3} \longrightarrow \frac{z - 1}{z - 3} \cdot \frac{2 - 3}{2 - 1} = -\frac{z - 1}{z - 3} = \begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix} (z)$$

Thus, the composition of the maps $1, 2, 3$ to $0, 1, \infty$ and then to z_1, z_2, z_3 is

$$z \longrightarrow \begin{pmatrix} -z_3 & z_1 \\ -1 & 1 \end{pmatrix} \Big(\frac{z_2 - z_1}{z_2 - z_3} \cdot \Big(-\frac{z - 1}{z - 3} \Big) \Big) \qquad \text{(sends } 1, 2, 3 \text{ to } z_1, z_2, z_3)$$

So-called simplification is most likely misguided. ///

[07.2] Exhibit a linear fractional transformation mapping the circle $|z| = 1$ to the line $\mathrm{Re}(z) = \mathrm{Im}(z)$.

Use the fact that linear fractional transformations preserve the collection of lines-and-circles, and that a line-or-circle is determined by three points on it, so tracking three points suffices to determine the image. The Cayley map $z \to \dfrac{z + i}{iz + 1}$ fixes ± 1, and sends $i \to \infty$, so maps the unit circle to the real line. Then rotate by $e^{i\pi/4}$. Altogether, this is

$$z \longrightarrow e^{i\pi/4} \cdot \frac{z + i}{iz + 1} = \frac{e^{i\pi/4} z + e^{i \cdot 5\pi/4}}{iz + 1} = \begin{pmatrix} e^{i\pi/4} & e^{i \cdot 5\pi/4} \\ i & 1 \end{pmatrix} (z)$$

mapping the unit circle to the diagonal. ///

[07.3] Exhibit a linear fractional transformation stabilizing the (open) upper half-plane $\mathfrak{H}$ and mapping i to $2+i$.

Granting that $SL_2(\mathbb{R})$ stabilizes $\mathfrak{H}$, and noting the general possibility

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} (i) \;=\; x + iy$$

we simply have

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} (i) \;=\; 2+i$$

as desired. ///

[07.4] Given $0 < t < 1$, exhibit a linear fractional transformation stabilizing the open unit disk, and mapping 0 to t.

Grant that the standard $SU(1,1)$ stabilizes the open unit disk, and that $\begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}$ is in $SU(1,1)$. Rather than try to solve equations involving hyperbolic functions, observe that for $v = \cosh u > 1$,

$$\begin{pmatrix} v & \sqrt{v^2-1} \\ \sqrt{v^2-1} & v \end{pmatrix}$$

is in $SU(1,1)$. It maps $0 \to \frac{\sqrt{v^2-1}}{v}$. Thus, solve for v in

$$\frac{\sqrt{v^2-1}}{v} \;=\; t \qquad\qquad \text{(solve for } v\text{)}$$

Multiply through by v, and square:

$$v^2 - 1 \;=\; v^2 \cdot t^2$$

or $(1-t^2)v^2 = 1$ and then $v = 1/\sqrt{1-t^2}$. Then

$$\begin{pmatrix} v & \sqrt{v^2-1} \\ \sqrt{v^2-1} & v \end{pmatrix} \;=\; \begin{pmatrix} \frac{1}{\sqrt{1-t^2}} & \frac{t}{\sqrt{1-t^2}} \\ \frac{t}{\sqrt{1-t^2}} & \frac{1}{\sqrt{1-t^2}} \end{pmatrix} \qquad\qquad \text{(maps 0 to } t\text{)}$$

as desired. ///

[07.5] Exhibit a conformal map of the sector $\{re^{i\theta} : r > 0,\ 0 < \theta < \frac{\pi}{4}\}$ to the unit disk.

First, note that the eighth-power map does not quite accomplish this, since the image of this open sector under the eighth power map omits the real interval $[0,1)$. Instead, the fourth power map $z \to z^4$ does send this sector to the open upper half-plane $\mathfrak{H}$, and then the inverse Cayley map sends $\mathfrak{H}$ to the open unit disk. Thus, $z \to \frac{z^4 - i}{-iz^4 + 1}$ maps the given sector to the open unit disk.

[07.6] Exhibit a conformal map from the strip $\{z = x+iy : c < ax + by < c'\}$ to the crescent

$$\Omega \;=\; \{z : |z| < 1,\ |z - \tfrac{1}{2}| > \tfrac{1}{2}\}$$

Both resgions are examples of *degenerate bi-gons*, namely, where the vertices are *not* distinct points, and, necessarily, the angles at the vertices are 0.

Perhaps it's easier to go in the opposite direction, since it's easier to adjust *strips* by rotations and dilations than to adjust *crescents* by linear fractional transformations stabilizing the outer circle, for example. Thus, map the single vertex $z_1 = 1$ of the crescent to ∞, by $z \to \frac{1}{z-1}$. The image of the outer circle is determined by tracking two more points on it, for example $\pm i$, and the image of the inner by tracking two points on it, for example, 0 and $\frac{1+i}{2}$. That is, the image of the outer circle is the straight line through $\frac{1}{i-1} = \frac{-i-1}{2}$ and $\frac{1}{-i-1} = \frac{i-1}{2}$, while the image of the inner circle is the straight line through $\frac{1}{-1} = -1$ and

$$\frac{1}{\frac{1+i}{2} - 1} \;=\; \frac{2}{1+i-2} \;=\; \frac{2}{i-1} \;=\; -i-1$$

That is, the image of the outer circle is the *vertical* line through $-\frac{1}{2}$, and the image of the inner circle is the vertical line through -1. Thus, the image of the crescent under $z \to \frac{1}{z-1}$ is the strip $\{z : -1 < \mathrm{Re}(z) < -\frac{1}{2}\}$.

Meanwhile, a relation $\{z = x + iy : c < ax + by < c'\}$ with real parameters a, b, c, c' can be rewritten as

$$\{z \;:\; c < \mathrm{Re}\Big(z \cdot (a - ib)\Big) < c'\} \;=\; (a - ib)^{-1} \cdot \{z \;:\; c < \mathrm{Re}(z) < c'\}$$

Further *real* translation and dilation can map any vertical strip to any other:

$$\{z \;:\; c < \mathrm{Re}(z) < c'\} \;=\; c + \{z \;:\; 0 < \mathrm{Re}(z) < c' - c\} \;=\; (c' - c) \cdot \Big(c + \{z \;:\; 0 < \mathrm{Re}(z) < 1\}\Big)$$

In the case at hand, first map by $z \to \frac{1}{z-1}$ to the strip $-1 < \mathrm{Re}(z) < -\frac{1}{2}$, then by $z \to z+1$ to $0 < \mathrm{Re}(z) < \frac{1}{2}$, then by $z \to z/2(c' - c)$ to $0 < \mathrm{Re}(z) < c' - c$, then by $z \to z + c$ to $c < \mathrm{Re}(z) < c'$, then by $z \to (a - bi)^{-1} z$ to $c < ax + by < c'$. ///

[07.7] Let holomorphic $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be 2-to-1. Show that there are two linear fractional transformations α, β such that $\alpha \circ f \circ \beta$ is the map $z \to z^2$.

The 2-to-1 property surely counts *multiplicities*.

We have shown that all holomorphic maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ are *rational* maps $f(z) = P(z)/Q(z)$ with polynomials P, Q. Without loss of generality P, Q are relatively prime in the principal ideal domain $\mathbb{C}[X]$. Certainly Q is not identically 0. If the degree of P is greater than 2, then (counting multiplicities) more than 2 points map to 0, contradiction. Similarly, if the degree of Q is more than 2, then more than 2 points map to ∞, contradiction.

Let $P(z) = az^2 + bz + c$ and $Q(z) = Az^2 + Bz + C$. Not both a, A can be 0, or else this is a linear fractional transformation, and is not 2-to-1. Post-composing with $z \to 1/z$ if necessary, we can suppose that $A \neq 0$. Then post-compose with a translation to make $a = 0$. This will simplify the algebra. Then

$$(P/Q)'(z) \;=\; \frac{b(Az^2 + Bz + C) - (bz + c)(2Az + B)}{Q^2(z)}$$

The numerator is

$$(bA)z^2 + (bB)z + bC - (2bA)z^2 - (bB + 2cA)z - cB \;=\; (-bA)z^2 + (bB - bB - 2cA)z + (bC - cB)$$

$$= \; (-bA)z^2 + 2(-cA)z + (bC - cB)$$

This has at least one zero unless the coefficients of z^2 and z are both 0, which, since $A \neq 0$, would require that $P(z) = 0$, contradiction.

Thus, there is a zero z_o of the numerator. Then

$$\Big(\frac{P(z)}{Q(z)} - \frac{P(z_o)}{Q(z_o)}\Big)' \;=\; 0$$

so z_o is a *double* zero of $P/Q - P(z_o)/Q(z_o)$, that is, P/Q takes the value $P(z_o)/Q(z_o)$ with multiplicity two at z_o. Pre-composing and post-composing with translations, without loss of generality $z_o = 0$ and $P(z_o)/Q(z_o) = 0$. This reduces to the form $z \to z^2/Q(z)$ with $Q(z) = Az^2 + Bz + C$ with $A \neq 0$ and $C \neq 0$.

Post-composing with $z \to 1/z$, we can consider $f(z) = Q(z)/z^2$, and by post-composing with a translation, $Q(z) = az + b$. If $a = 0$, then $f(z) = b/z^2$, and post-composing with $z \to 1/z$ (and with a dilation) gives $f(z) = z^2$.

With $a \neq 0$, and with $b \neq 0$ to avoid cancellation and reduction to a linear fractional transformation (which would not be 2-to-1), computing a derivative again,

$$f'(z) \;=\; \frac{az^2 - (az+b)2z}{z^4} \;=\; \frac{-az^2 - 2bz}{z^4} \;=\; \frac{-az - 2b}{z^3}$$

This has a zero at $z_o = -2b/a \neq 0$. Thus, $\frac{Q(z)}{z^2} - \frac{Q(z_o)}{z_o^2}$ assumes the value 0 with multiplicity 2 at $z_o \neq 0$. Up to a constant, it is $(z - z_o)^2/z^2 = \Big(\frac{z - z_o}{z}\Big)^2$. Pre-composing with the inverse to $z \to (z - z_o)/z$ makes this $f(z) = z^2$.

[07.8] What happens to the zero set of $z \to e^{2\pi i z}$ under the perturbation $z \longrightarrow e^{2\pi i z} - hz$ for small h?

There are obvious variants of this, for example, $z \to e^{2\pi i z} - 1 - hz$ really does have infinitely-many zeros at $h = 0$, and as h moves away from 0 each one of these is (locally) a holomorphic function of h, by the holomorphic inverse function theorem.

One reason to mention $z \to e^{2\pi i z} - hz$ is to exhibit a seemingly discontinuous phenomenon, perhaps intuitively opposite to the continuity of zeros *already in existence*: at $h = 0$ the function $z \to e^{2\pi i z}$ it has *no* zeros whatsoever, while for every non-zero h the function $z \to e^{2\pi i z} - hz$ it suddenly has *infinitely-many* zeros.

Despite the abrupt change from $h = 0$ to $h \neq 0$, for the latter we can use the *argument principle*: estimate the net change in the argument of $f(z)$ around a large rectangle to estimate 2π times the number of zeros inside the box. Naturally, we adjust the box slightly so that no zeros are exactly on its edges.

Along the bottom edge of the box, $|e^{2\pi i z}| = e^{-2\pi \mathrm{Im}\,(z)}$ tends to be larger than $|h| \cdot |z|$ simply because exponentials grow faster than polynomials. The limit of this is the possibility that the rectangle is very wide in comparison to its height, so that $x = \mathrm{Re}(z)$ becomes large enough so that $e^{-2\pi y} < |h| \cdot |z|$. Excluding the latter possibility, the argument of $e^{2\pi i z} - hz$ is within $\pi/2$ of the argument of $e^{2\pi i z}$, which changes by 2π times the width of the box.

Along the top edge, $|e^{2\pi i z}| = e^{-2\pi y}$ tends to be smaller than $|h| \cdot |z|$, so the argument of $e^{2\pi i z} - hz$ is within $\pi/2$ of the argument of $h \cdot z$, which changes by less than π along the top of the box.

Along left or right vertical edges, use the idea that the net change in argument along a given curve is at most $2\pi \cdot (q+1)$ where q is the number of zeros of the real part of $e^{2\pi i z} - hz$ along the curve. The real part of the function here is $e^{-2\pi y} \cos x - \mathrm{Re}\,(h)x - \mathrm{Im}\,(h)y$. For simplicity, adjust the location of the vertical sides of the box by a small amount so that $\cos x = 0$. Then the real part vanishes at most once, so the total change in argument is at most $2\pi \cdot 2 = O(1)$, using Landau's big-O notation.

Thus, on a sufficiently large (depending on h) box with vertices $\pm T \pm iT$, adjusting the location of the vertical sides slightly, the number of zeros inside is

$$\frac{1}{2\pi}\Big(\text{change in arg over top, bottom, left, right}\Big) \;=\; \frac{1}{2\pi}\Big(2\pi \cdot 2T + O(1)\Big) \;=\; 2T + O(1)$$

In fact, examining the estimates, the top edge can be much lower, and the left and right sides can be pushed out quite a lot, and the same type of formula applies. ///
