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Complex analysis examples discussion 08

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[08.1] Check that the Euclidean Laplacian $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ on $\mathbb{R}^n$ is *rotation-invariant*, in the following sense. A *rotation* is a linear map $g : \mathbb{R}^n \to \mathbb{R}^n$ preserving the usual inner product $\langle x, y\rangle = \sum_i x_i y_i$, and preserving orientations (so $\det g = 1$, rather than -1). The asserted rotation-invariance is

$$\Delta(f \circ g) \;=\; (\Delta f) \circ g \qquad \text{(for twice-differentiable } f \text{ and rotation } g)$$

(In fact, Δ is also preserved by *reflections*, which are orientation-reversing, so the determinant condition can be safely ignored.)

The n-by-n real matrix g is a rotation-or-reflection when $g^\top g = 1_n$. Element-wise, this is

$$\delta_{ik} \;=\; \sum_j (g^\top)_{ij} g_{jk} \;=\; \sum_j g_{ji} g_{jk} \qquad \text{(with Kronecker delta } \delta_{ik})$$

Elements $x \in \mathbb{R}^n$ can be expressed as either row vectors or column vectors, with g acting either by right multiplication or left, respectively, without affecting the conclusion. We choose *row* vectors and *right* multiplication:

$$\Delta(f \circ g)(x) \;=\; \sum_\ell (\frac{\partial}{\partial x_\ell})^2 f(\ldots, \sum_i x_i g_{ij}, \ldots) \;=\; \sum_\ell \frac{\partial}{\partial x_\ell} \sum_s g_{\ell s} f_s(\ldots, \sum_j x_i g_{ij}, \ldots)$$

where f_s is the partial derivative of f with respect to its s^{th} argument. Taking the next derivative gives

$$\sum_\ell \sum_{s,t} g_{\ell s} g_{\ell t} f_{st}(\ldots, \sum_i x_i g_{ij}, \ldots)$$

Interchange the order of the sums and use $\sum_\ell g_{\ell s} g_{\ell t} = \delta_{st}$:

$$\sum_s f_{ss}(\ldots, \sum_i x_i g_{ij}, \ldots) \;=\; (\Delta f)(xg) \;=\; ((\Delta f) \circ g)(x)$$

as desired. ///

[08.2] Check that for harmonic h and holomorphic f, the composition $h \circ f$ is invariably harmonic, while $f \circ h$ need not be. (Yes, much of the issue is suitable formulation of the computation.)

First the easy part: with holomorphic $f(z) = z^2$ and harmonic $h(x + iy) = y$,

$$f(h(x + iy)) \;=\; f(y) \;=\; y^2$$

and $\Delta y^2 = 2 \neq 0$, so $f \circ h$ is not harmonic.

To prove that $h \circ f$ is harmonic, write h as a function of the real and imaginary parts of a complex number, and write $f(x + iy) = u(x, y) + iv(x, y)$. Then

$$\Delta(h(u(x, y), v(x, y))) \;=\; \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) h(u(x, y), v(x, y)))$$

$$= \; \frac{\partial}{\partial x}\Big(h_1(u, v)u_x + h_2(u, v)v_x\Big) + \frac{\partial}{\partial y}\Big(h_1(u, v)u_y + h_2(u, v)v_y\Big)$$

$$= \; \Big(h_{11}u_x^2 + (h_{12} + h_{21})u_x v_x + h_1 u_{xx} + h_{22}v_x^2 + h_2 v_{xx}\Big) + \Big(h_{11}u_y^2 + (h_{12} + h_{21})u_y v_y + h_1 u_{yy} + h_{22}v_y^2 + h_2 v_{yy}\Big)$$

$$= \; h_{11}(u_x^2 + u_y^2) + (h_{12} + h_{21})(u_x v_x + u_y v_y) + h_1(u_{xx} + u_{yy}) + h_{22}(v_x^2 + v_y^2) + h_2(v_{xx} + v_{yy})$$

The real and imaginary parts u, v of f are themselves harmonic, so the h_1 and h_2 terms vanish, leaving

$$h_{11}(u_x^2+u_y^2)+(h_{12}+h_{21})(u_xv_x+u_yv_y)+h_{22}(v_x^2+v_y^2)$$

In terms of real and imaginary parts, the Cauchy-Riemann equation

$$\Big(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\Big)(u+iv) \;=\; 0$$

becomes

$$u_x+v_y \;=\; 0 \qquad\qquad \text{and} \qquad\qquad -u_y+v_x \;=\; 0$$

Thus, the coefficient of $h_{12}+h_{21}$ is

$$u_xv_x+u_yv_y \;=\; u_xu_y+u_y(-u_x) \;=\; 0$$

and

$$h_{11}(u_x^2+u_y^2)+h_{22}(v_x^2+v_y^2) \;=\; h_{11}(u_x^2+v_x^2)+h_{22}(v_x^2+(-u_x)^2) \;=\; (h_{11}+h_{22})\cdot(u_x^2+v_x^2) \;=\; 0\cdot(u_x^2+v_x^2)$$

so $h\circ f$ is holomorphic. ///

[08.3] Show that every harmonic function u on an annulus $r<|z|<R$ is of the form

$$u(z) \;=\; a_0+b_0\log|z|+\sum_{0\neq n\in\mathbb{Z}}\big(a_nz^n+b_n\overline{z}^n\big)$$

for constants a_i, b_i.

Use polar coordinates $z=re^{i\theta}$ on $0<|z|<1$, and express u as a Fourier series in θ with coefficients that are functions of r:

$$u(re^{i\theta}) \;=\; \sum_{n\in\mathbb{Z}} c_n(r)\,e^{in\theta}$$

In polar coordinates, with $u(re^{i\theta})=f(r,\theta)$, with $r=\sqrt{x^2+y^2}$ and $\theta=\arctan\frac{y}{x}$,

$$r_x \;=\; \frac{x}{r} \qquad r_y \;=\; \frac{y}{r} \qquad \theta_x \;=\; \frac{-\frac{y}{x^2}}{1+(\frac{y}{x})^2} \;=\; \frac{-y}{r^2} \qquad \theta_y \;=\; \frac{\frac{1}{x}}{1+(\frac{y}{x})^2} \;=\; \frac{x}{r^2}$$

and the Laplacian is

$$\begin{aligned}\Delta u \;=\; =\; &\frac{\partial}{\partial x}\Big(f_rr_x-f_\theta\theta_x\Big)+\frac{\partial}{\partial y}\Big(f_rr_y+f_\theta\theta_y\Big) \;=\; \frac{\partial}{\partial x}\Big(f_r\frac{x}{r}-f_\theta\frac{y}{r^2}\Big)+\frac{\partial}{\partial y}\Big(f_r\frac{y}{r}+f_\theta\frac{x}{r^2}\Big)\\ =\;&\Big(f_{rr}\big(\frac{x}{r}\big)^2+f_r\big(\frac{1}{r}-\frac{x^2}{r^3}\big)-(f_{r\theta}+f_{\theta r})\frac{xy}{r^3}+f_{\theta\theta}\big(\frac{y}{r^2}\big)^2-f_\theta\frac{2xy}{r^4}\Big)\\ &+\Big(f_{rr}\big(\frac{y}{r}\big)^2+f_r\big(\frac{1}{r}-\frac{y^2}{r^3}\big)+(f_{r\theta}+f_{\theta r})\frac{xy}{r^3}+f_{\theta\theta}\big(\frac{x}{r^2}\big)^2+f_\theta\frac{2xy}{r^4}\Big)\\ =\;& f_{rr}+\frac{f_r}{r}+\frac{f_{\theta\theta}}{r^2} \;=\; \Big(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}++\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)f\end{aligned}$$

Applying this to the Fourier expansion, differentiating termwise,

$$0 \;=\; \Delta f(r,\theta) \;=\; \sum_n \Delta(c_n(r)e^{in\theta}) \;=\; \Big(c_n''+\frac{1}{r}c_n'+\frac{1}{r^2}c_n(in)^2\Big)\,e^{in\theta}$$

By *uniqueness* of Fourier expansions,

$$c_n'' + \frac{1}{r}c_n' - \frac{n^2}{r^2}c_n \;=\; 0$$

This equation is of *Euler type*, with indicial equation

$$\alpha(\alpha-1) + \alpha - n^2 \;=\; 0$$

with solutions $\alpha = \pm n$. For $n = 0$, the root $\alpha = 0$ is doubled, and solutions of the differential equation are $r^0 = 1$ and $r^0 \cdot \log r = \log r$. For $n \neq 0$, the solutions are r^n and r^{-n}.

Translating back to z and $\overline{z}$ coordinates, we obtain the indicated expansion. ///

[08.4] Show that a harmonic function u on $0 < |z| < 1$ such that

$$\int_{0<x^2+y^2<1} |u(x+iy)|^2 \; dx\, dy \;<\; \infty$$

is of the form $u(x+iy) = v(x+iy) + c\log|z|$ for v harmonic on the disk $|z| < 1$, for some constant c.

There is a Fourier expansion

$$u(z) \;=\; \sum_{n\in\mathbb{Z}} c_n(r)\, e^{in\theta}$$

and the integral of $|u|^2$ over the punctured disk is

$$\int_{0<r<1} |u|^2 \;=\; \sum_{m,n}\int_0^1 c_m(r)\,\overline{c}_n(r)\Big(\int_0^{2\pi} e^{im\theta}e^{-in\theta}\; d\theta\Big) r\, dr \;=\; 2\pi\sum_n \int_0^1 |c_n(r)|^2 r\, dr$$

by orthogonality of distinct exponentials. From the previous example, the 0^{th} Fourier coefficient is a linear combination $a_0 + b_0 \log r$, and

$$\int_0^1 \Big|a_0 + b_0\log r\Big|^2 \; r\, dr \;<\; \infty \qquad\qquad \text{(for arbitrary } a_0, b_0\text{)}$$

In contrast, for $n > 0$,

$$\int_0^1 \Big|a_n r^n + b_n r^{-n}\Big|^2 \; r\, dr \;=\; \int_0^1 \Big(|a_n|^2 r^{2n} + (a_n\bar{b}_n + \bar{a}_n b_n) + |b_n|^2 r^{-2n}\Big)\; r\, dr$$

The first two summands have finite integrals, but $\int_0^1 r^{-2n}\; r\, dr = +\infty$. Thus, $b_n = 0$. That is, apart from the $\log r$ term, the only non-zero coefficients in the Fourier expansion give terms $z^n = r^n e^{-n\theta}$ and $\overline{z}^n = r^n e^{-in\theta}$ with $n \geq 0$. A sum of such terms is harmonic on the whole disk. ///

[08.5] Define f on the unit circle by $f(e^{i\theta}) = \theta^2$, for $-\pi < \theta < \pi$. Find a harmonic function u on the open disk whose boundary values are f.

There are several ways to think about this. One is to determine the Fourier expansion of the boundary-value function, make the obvious extension to a sum of power series in z and $\overline{z}$ as in that sort of derivation of the Poisson kernel, and then presumably sum the resulting series to an elementary function. Yes, this amounts to re-doing part of the discussion of the Poisson kernel, but may be a reasonable choice in circumstances where the integral against the Poisson kernel cannot be expressed in elementary closed form.

The 0^{th} Fourier coefficient of the function $F(\theta) \to \theta^2$ on $[-\pi, \pi]$ is

$$\widehat{F}(0) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta)\; d\theta \;=\; \frac{\pi^3/3}{2\pi} \;=\; \frac{\pi^2}{6}$$

For $n \neq 0$, the n^{th} Fourier coefficient is computed by integrating by parts:

$$\widehat{F}(n) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta^2 \cdot e^{-in\theta}\,d\theta \;=\; \frac{1}{2\pi}\Big[\theta^2 \cdot \frac{e^{-in\theta}}{-in}\Big]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\theta \cdot \frac{e^{-in\theta}}{-in}\,d\theta \;=\; \frac{1}{\pi i n}\int_{-\pi}^{\pi} \theta \cdot e^{-in\theta}\,d\theta$$

$$= \frac{1}{\pi i n}\Big[\theta \cdot \frac{e^{-in\theta}}{-in}\Big]_{-\pi}^{\pi} - \frac{1}{\pi i n}\int_{-\pi}^{\pi} \frac{e^{-in\theta}}{-in}\,d\theta \;=\; \frac{(-1)^n}{n^2}$$

Thus,

$$f(e^{i\theta}) \;=\; F(\theta) \;=\; \frac{\pi^2}{6} + \sum_{n\neq 0} \frac{(-1)^n}{n^2} e^{in\theta}$$

As in the Fourier series treatment of the Dirichlet problem, extend the positive-index exponentials to powers of z and the negative exponentials to powers of $\overline{z}$: put

$$u(z) \;=\; \frac{\pi^2}{6} + \sum_{n\geq 1} \frac{(-1)^n}{n^2} z^n + \sum_{n\geq 1} \frac{(-1)^n}{n^2} \overline{z}^n$$

This function seems to have no simpler expression, although there is the related elementary identity

$$\frac{d}{dx}\Big(\sum_{n\geq 1} \frac{(-1)^n}{n^2} x^n\Big) \;=\; -1 + \frac{z}{2} - \frac{z^2}{3} - \ldots \;=\; -\frac{1}{x}\log(1+x)$$

[08.6] *(Euler-type equations of second order)* An ordinary differential equation of the form

$$u'' + \frac{b}{x}u' + \frac{c}{x^2}u \;=\; 0$$

with constants b, c is said to be of *Euler type*. Show that it has solutions x^α and x^β where α, β are solutions of the **auxiliary equation**

$$\lambda(\lambda - 1) + b\lambda + c \;=\; 0$$

Show that $x^\alpha \log x$ is the second solution if the root of the auxiliary equation is *double*, i.e., if $\alpha = \beta$. Use the Mean Value Theorem to genuinely prove that there are no other solutions.

We choose to consider these differential equations on $(0, +\infty)$, so that complex powers x^α are unambiguous.

Among various ways to discuss Euler-type equations, an approach that scales up to higher-degree versions

$$x^n u^{(n)} + c_{n-1}x^{n-1}u^{(n-1)} + \ldots + c_2 x^2 u'' + c_1 x u' + c_0 u \;=\; 0$$

observes that differential operators $x^\ell \frac{\partial^\ell}{\partial x^\ell}$ are all polynomials in the single operator $x\frac{\partial}{\partial x}$, and *factors* the differential operator

$$x^n \frac{\partial^n}{\partial x^n} + c_{n-1}x^{n-1}\frac{\partial^{n-1}}{\partial x^{n-1}} + \ldots + +c_1 x\frac{\partial}{\partial x} + c_0 \;=\; \Big(x\frac{\partial}{\partial x} - \alpha_1\Big)\Big(x\frac{\partial}{\partial x} - \alpha_2\Big)\ldots\Big(x\frac{\partial}{\partial x} - \alpha_n\Big)$$

for constants α_j. We solve the equation

$$\Big(x\frac{\partial}{\partial x} - \alpha_1\Big)\Big(x\frac{\partial}{\partial x} - \alpha_2\Big)\ldots\Big(x\frac{\partial}{\partial x} - \alpha_n\Big)u \;=\; 0$$

in steps, first solving $\big(x\frac{\partial}{\partial x} - \alpha_1\big)u_1 = 0$, then $\big(x\frac{\partial}{\partial x} - \alpha_2\big)u_2 = u_1$, then $\big(x\frac{\partial}{\partial x} - \alpha_3\big)u_3 = u_2$, and so on.

Solving $xu' - \alpha u = 0$ is easy: $u(x)$ is a constant multiple of x^α. Uniqueness is proven via the mean value theorem: for another solution $u(x)$, with $v(x) = u(x)/x^\alpha$,

$$x(x^\alpha v)' - \alpha(x^\alpha v) \;=\; 0$$

gives

$$0 \;=\; x(\alpha x^{\alpha-1} v + x^\alpha v') - \alpha x^\alpha v \;=\; x^{\alpha+1} v'$$

That is, $v' = 0$ and v is a constant, proving uniqueness.

Continue by induction: $xu' - \alpha u = x^\beta$ with $\alpha \neq \beta$ has solution $x^\beta/(\beta - \alpha)$, and uniqueness for the associated homogeneous equation gives uniqueness. Thus, when the α_i are all different, all solutions are linear combinations of x^{α_i}.

To treat multiple roots, a general observation is helpful: for any (linear) differential operator D and function u satisfying $(D - \lambda)u = 0$, with u depending of course on λ, differentiation in λ gives

$$\frac{-u + (D - \lambda)\partial u}{\partial \lambda \;=\; 0}$$

or

$$\frac{(D - \lambda)\partial u}{\partial \lambda \;=\; u}$$

Repeating,

$$\frac{(D - \lambda)\partial^2 u}{\partial \lambda^2 \;=\; \frac{\partial u}{\partial \lambda}}$$

and so on. In the case at hand, the n^{th} derivative of x^α with respect to the eigenvalue α is $x^\alpha \cdot \log^n x$, and

$$(x\frac{\partial}{\partial x} - \alpha)(x^\alpha \cdot \log^n x) \;=\; x^\alpha \cdot \log^{n-1} x$$

Thus, with roots α with multiplicities $\nu = \nu_\alpha$, all solutions are linear combinations of x^α, $x^\alpha \log x$, $x^\alpha \log^2 x$, $\ldots$, $x^\alpha \log^{\nu-1} x$. ///

[08.7] *(Rotationally invariant harmonic functions in $\mathbb{R}^n$)* For f twice-differentiable on $\mathbb{R}^n$, expressible as a (twice-differentiable) function of the *radius* r alone (at least away from 0), say f is *spherically symmetric* or *rotationally invariant*. (This could also be formulated as invariance under the action of the orthogonal group by rotations). Show that

$$\Delta\, f \;=\; f'' + \frac{n-1}{r} f'$$

(This is of Euler type). On $\mathbb{R}^n - \{0\}$, find two linearly independent harmonic functions.

Compute directly, with $r = \sqrt{x_1^2 + \ldots + x_n}$,

$$\Delta f(r) \;=\; \sum_i \frac{\partial^2}{\partial x_i^2} f(r) \;=\; \sum_i \frac{\partial}{\partial x_i}\Big(\frac{x_i}{r} \cdot f'(r)\Big) \;=\; \sum_i \frac{\partial}{\partial x_i}\Big(\frac{1}{r} f' - \frac{x_i^2}{r^3} f' + \big(\frac{x_i}{r}\big)^2 f''\Big) \;=\; f'' + \frac{n-1}{r} f'$$

The indicial equation is

$$\lambda(\lambda - 1) + (n-1)\lambda \;=\; 0$$

so $\lambda = 0, 2-n$. For $n \neq 2$, the roots are distinct, giving linearly independent solutions $1, x^{2-n}$. For $n = 2$, the root is doubled, giving linearly independent solutions $1, \log x$.

[08.8] The Fourier expansion

$$\delta(\theta) \;=\; \sum_{n\in\mathbb{Z}} e^{in\theta} \;=\; \sum_{n\in\mathbb{Z}} \widehat{\delta}(n)\, e^{in\theta} \qquad\qquad (\text{with } \widehat{\delta}(n) = 1 \text{ for all } n \in \mathbb{Z})$$

certainly does not converge *pointwise*, but does make sense as the expansion of the periodic Dirac δ, sometimes called *Dirac comb* function on $\mathbb{R}/2\pi\mathbb{Z}$, in the following sense. The *Plancherel identity*

$$\langle u, v\rangle \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\,\overline{v(\theta)}\,d\theta \;=\; \sum_{n\in\mathbb{Z}} \widehat{u}(n)\cdot\overline{\widehat{v}(n)} \qquad\qquad (\text{for } u,v\in L^2(S^1))$$

$L^2(S^1)\times L^2(S^1)\to\mathbb{C}$ can be restricted in the first argument and extended in the second, so that for *smooth* $u(\theta)=\sum_{n\in\mathbb{Z}}\widehat{u}(n)\,e^{in\theta}$, pairing against δ correctly evaluates u at $\theta=0$:

$$u(0) \;=\; \sum_n \widehat{u}(n)\,e^{in\cdot 0} \;=\; \sum_{n\in\mathbb{Z}}\widehat{u}(n)\cdot 1 \;=\; \sum_{n\in\mathbb{Z}}\widehat{u}(n)\cdot\overline{\widehat{\delta}(n)} \;=\; \langle u,\delta\rangle$$

Identifying the circle with the boundary $\{z : |z|=1\}$ of the disk $\{z : |z|<1\}$, determine the harmonic function on the disk whose boundary value function is the periodic Dirac δ.

Again following the path in the Fourier series derivation of the Poisson kernel, replace positive-index Fourier terms by powers of z, and negative-index Fourier terms by powers of $\overline{z}$, producing

$$u(z) \;=\; 1+\sum_{n\geq 1} z^n + \sum_{n\geq 1}\overline{z}^n \;=\; 1+\frac{z}{1-z}+\frac{\overline{z}}{1-\overline{z}} \;=\; \frac{(1-z-\overline{z}+|z|^2)+z(1-\overline{z})+\overline{z}(1-z)}{(1-z)(1-\overline{z})} \;=\; \frac{1-|z|^2}{|1-z|^2}$$

Yes, this is the Poisson kernel $P(e^{i\theta},z)$ at $\theta=0$. ///
