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Complex analysis examples 09

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http://www.math.umn.edu/˜garrett/m/complex/examples_2014-15/cx_discussion_09.pdf]

[09.1] Prove that

$$\lim_{N\to+\infty}\prod_{n=1}^{N}(1+\frac{1}{n}) \;=\; +\infty \qquad\qquad \text{and} \qquad\qquad \lim_{N\to+\infty}\prod_{n=2}^{N}(1-\frac{1}{n}) \;=\; 0$$

In the first case, by multiplying out and dropping some positive terms (an induction, if you want to be formal),

$$\prod_{n=1}^{N}(1+\frac{1}{n}) \;=\; 1+\Big(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{N}\Big)+\Big(\frac{1}{1\cdot 2}+\frac{1}{1\cdot 3}+\ldots\Big)+\ldots \;\geq\; \frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{N}$$

$$\geq\; \int_1^2 \frac{1}{t}dt+\int_2^3\frac{1}{t}\,dt+\ldots+\int_N^{N+1}\frac{1}{t}\,dt \;=\; \int_1^{N+1}\frac{1}{t}\,dt \;=\; \log(N+1) \;\longrightarrow\; +\infty$$

since $\frac{1}{\ell}\geq\int_\ell^{\ell+1}\frac{1}{t}\,dt$ by the monotonicity of $t\to\frac{1}{t}$. ///

For the second product, take logarithms and Taylor expansions: use

$$\log(1-t) \;=\; -\Big(t+\frac{t}{2}+\frac{t}{3}+\ldots\Big) \;\leq\; -t \qquad\qquad (\text{for } 0<t<1)$$

Then

$$\log\prod_{n=2}^{N}(1-\frac{1}{n}) \;=\; \sum_{n=2}^{N}\log(1-\frac{1}{n}) \;\leq\; \sum_{n=2}^{N}-\frac{1}{n}) \;\leq\; -\log(N-1) \;\longrightarrow\; -\infty$$

For a sequence t_n of positive reals, $\log t_n\to-\infty$ implies $t_n\to 0$. ///

[09.2] Following Euler, show that $\sum_{p\text{ prime}}\frac{1}{p}$ diverges, by using the Euler product expansion of $\zeta(s)$ and considering $s\to 1^+$ along the real axis.

For real $s>1$, in addition to the Euler product, we also need that $\sum_{\ell\geq 2,\,p}\frac{1}{\ell p^{\ell s}}$ with p summed over primes, *converges*, by estimating primes by natural numbers ≥ 2:

$$\sum_{\ell\geq 2,\,p}\frac{1}{\ell p^{\ell s}} \;\leq\; \sum_{\ell\geq 2,\,n\geq 2}\frac{1}{\ell n^{\ell s}} \;\leq\; \sum_{\ell\geq 2}\frac{1}{\ell}\int_1^\infty\frac{1}{t^{\ell s}}\,dt \;=\; \sum_{\ell\geq 2}\frac{1}{\ell}\cdot\frac{1}{\ell s-1} \;<\; +\infty$$

uniformly in $s>1$. Letting C be that finite bound,

$$C+\sum_p\frac{1}{p^s} \;\geq\; \sum_p\Big(\frac{1}{p^s}+\frac{1}{2p^{2s}}+\frac{1}{3p^{3s}}+\ldots\Big) \;=\; \sum_p-\log(1-\frac{1}{p^s}) \;=\; \sum_p\log\frac{1}{1-\frac{1}{p^s}} \;=\; \log\prod_p\frac{1}{1-\frac{1}{p^s}}$$

$$=\; \log\zeta(s) \;\geq\; \log\Big(\sum_n\frac{1}{n^s}\Big) \;\geq\; \log\Big(\int_1^\infty\frac{1}{t^s}\,dt\Big) \;=\; \log(\frac{1}{s-1})$$

As $s\to 1^+$, the logarithm blows up, so the sum $\sum_p 1/p^s$ must be infinite. ///

[09.3] Prove that $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ does not vanish in $\mathrm{Re}(s) > 1$.

Let $\sigma = \mathrm{Re}(s)$, so $|p^s| = p^\sigma$, and

$$|1 - \frac{1}{p^s}| \;\le\; 1 + |\frac{1}{p^s}| \;=\; 1 + \frac{1}{p^\sigma} \qquad \text{and then} \qquad \frac{1}{|1-\frac{1}{p^s}|} \;\ge\; \frac{1}{1+\frac{1}{p^\sigma}}$$

Taking the logarithm of the Euler product, with $\sigma > 1$, using monotonicity of logarithm,

$$\log|\zeta(s)| \;=\; \log\Big|\prod_p \frac{1}{1-\frac{1}{p^s}}\Big| \;=\; \log\prod_p \frac{1}{|1-\frac{1}{p^s}|} \;=\; \sum_p \log\frac{1}{|1-\frac{1}{p^s}|} \;\ge\; \sum_p \log\frac{1}{1+\frac{1}{p^\sigma}}$$

$$= \sum_p -\log(1+\frac{1}{p^\sigma}) \;=\; \sum_p -\Big(\frac{1}{p^\sigma} - \frac{1}{2p^{2\sigma}} + \frac{1}{3p^{3\sigma}} - \ldots\Big) \;\ge\; -\sum_{\ell\ge1,\,p} \frac{1}{\ell\,p^{\ell\sigma}} \;\ge\; -\sum_{\ell\ge1,\,n\ge2} \frac{1}{\ell\,n^{\ell\sigma}}$$

$$\ge\; -\sum_{\ell\ge1} \frac{1}{\ell}\int_1^\infty \frac{1}{t^{\ell\sigma}}\,dt \;=\; -\sum_{\ell\ge1}\frac{1}{\ell}\cdot\frac{1}{\ell\sigma-1}$$

Thus, $\log|\zeta(s)| > -\infty$, so $\zeta(s) \ne 0$ there. ///

[09.4] Prove that $\Gamma(s)\cdot\Gamma(1-s) = \pi/\sin\pi s$, hence that $\Gamma(s)$ has no zeros, and $1/\Gamma(s)$ is entire.

Take $0 < \mathrm{Re}(s) < 1$, so that the Euler integrals for both $\Gamma(s)$ and $\Gamma(1-s)$ converge. Then

$$\Gamma(s)\cdot\Gamma(1-s) \;=\; \int_0^\infty\int_0^\infty e^{-t}t^s\cdot e^{-u}u^{1-s}\,\frac{dt}{t}\,\frac{du}{u}$$

Replacing t by tu gives

$$\int_0^\infty\int_0^\infty e^{-tu}t^s\,e^{-u}u^1\,\frac{dt}{t}\,\frac{du}{u} \;=\; \int_0^\infty\int_0^\infty e^{-tu}t^s\,e^{-u}\,\frac{dt}{t}\,du \;=\; \int_0^\infty\int_0^\infty e^{-(t+1)u}\,t^s\,\frac{dt}{t}\,du \;=\; \int_0^\infty \frac{t^{s-1}}{t+1}\,dt$$

This invites replacing the path from 0 to ∞ by the *Hankel contour* H_ε described as follows. Far to the right on the real line, start with the branch of t^{s-1} given by $(e^{2\pi i}t)^{s-1} = e^{2\pi i(s-1)}t^{s-1}$, integrate from $+\infty$ to $\varepsilon > 0$ along the real axis, clockwise around a circle of radius ε at 0, then back out to $+\infty$, now with the standard branch of t^{s-1}. For $\mathrm{Re}(s-1) > -1$ the integral around the little circle goes to 0 as $\varepsilon \to 0$. Thus,

$$\int_0^\infty \frac{t^{s-1}}{1+t}\,dt \;=\; \lim_{\varepsilon\to0}\frac{1}{1-e^{2\pi is(s-1)}}\int_{H_\varepsilon}\frac{t^{s-1}}{1+t}\,dt$$

The integral of this integrand over a large circle goes to 0 as the radius goes to $+\infty$, for $\mathrm{Re}(s-1) < 0$. Thus, this integral is equal to the limit as $R \to +\infty$ and $\varepsilon \to 0$ of the path integral over the Hankel/keyhole contour

from R to ε, from ε clockwise back to ε, from ε to R, from R counterclockwise to R

This integral is $2\pi i$ times the sum of the residues inside it, namely, that at $t = -1 = e^{\pi i}$. Thus, using $e^{\pi i} = -1$,

$$\Gamma(s)\Gamma(1-s) \;=\; \int_0^\infty \frac{t^{s-1}}{1+t}\,dt \;=\; \frac{2\pi i}{1-e^{2\pi i(s-1)}}\cdot(e^{\pi i})^{s-1} \;=\; \frac{2\pi i}{e^{-\pi i(s-1)}-e^{\pi i(s-1)}} \;=\; -\frac{2\pi i}{e^{-\pi is}-e^{\pi is}} \;=\; \frac{\pi}{\sin\pi s}$$

giving the asserted identity for $0 < \mathrm{Re}(s) < 1$, and then everywhere by the Identity Principle.

Next, since $\sin \pi s$ is *entire*, $\pi/\sin \pi s$ has no zeros, but has poles at all integers. Thus, if $\Gamma(s)$ had a zero s_o, by $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$, necessarily s_o is a pole of $\Gamma(1-s)$, to cancel in the product. From the identity $s\Gamma(s) = \Gamma(s+1)$, obtained by integration by parts, we saw that the poles of $\Gamma(s)$ are simple poles at non-positive integers, so the poles of $\Gamma(1-s)$ are simple poles at positive integers, and the only possible zeros of $\Gamma(s)$ would be at positive integers. We know this does not happen, for several possible reasons. For one, these values are factorials $\Gamma(n) = (n-1)!$. For another, $\pi/\sin \pi s$ does have poles at all integers, so the simple poles of $\Gamma(1-s)$ are not cancelled in the product $\Gamma(s)\Gamma(1-s)$. That is, $\Gamma(s)$ has no zeros, so $1/\Gamma(s)$ is entire. ///

[09.5] Prove that $\frac{1}{\Gamma(s)} = s\, e^{a+bs} \cdot \prod_{n=1}^{\infty}(1+\frac{s}{n})\, e^{-s/n}$ for some constants a, b.

We have shown that $1/\Gamma(s)$ is entire. We need to prove that its growth order is $\lambda < 2$, to invoke Hadamard's product theorem $h \le \lambda < h+1$. Then its genus is $h = 1$, so in its Weierstraß product

$$\frac{1}{\Gamma(s)} \;=\; e^{g(s)} \cdot s \prod_{n=1}^{\infty} \Big(1+\frac{s}{n}\Big)\, e^{\big(\frac{s}{-n}\big) + \big(\frac{s}{-n}\big)^2/2 + \ldots \pm + \big(\frac{s}{-n}\big)^{h_n}/h_n}$$

the entire function $g(s)$ is a polynomial of degree at most $h = 1$, and the compensating exponential factors are simply $e^{-s/n}$.

A simple form of the Stirling-Laplace asymptotic for $\Gamma(s)$ is

$$\lim_{|s|\to\infty} \frac{\Gamma(s)}{\sqrt{2\pi}\, e^{-s}\, s^{s-\frac{1}{2}}} \;=\; 1 \qquad\qquad (\text{in } \mathrm{Re}(s) \ge \delta > 0)$$

with fixed $\delta > 0$, so

$$\Big|\frac{1}{\Gamma(s)}\Big| \;\sim\; \frac{1}{\sqrt{2\pi}}\, e^{s}\, s^{\frac{1}{2}-s} \qquad\qquad (\text{in } \mathrm{Re}(s) \ge \delta > 0)$$

The e^s factor is of growth order 1. With $|s|$ bounded away from 0, $|\log s| \ll_\varepsilon |s|^\varepsilon$ for every $\varepsilon > 0$, so $\sqrt{s} \ll_{\delta,\varepsilon} e^{|s|^\varepsilon}$ in $\mathrm{Re}(s) \ge \delta > 0$. Letting $s = \sigma + it$, the subtlest part is

$$|s^s| \;=\; |e^{s\log s}| \;=\; e^{\mathrm{Re}\,(s\log s)} \;=\; e^{\sigma\cdot \mathrm{Re}\,(\log s) - t\cdot \mathrm{Im}\,(\log s)} \;\le\; e^{|s|\cdot|\log s|} \;\ll_{\delta,\varepsilon}\; e^{|s|^{1+\varepsilon}} \qquad\qquad (\text{for all } \varepsilon > 0)$$

where, of course, the implied constant depends on δ and ε. The product of functions of growth order 1 has growth order 1:

$$e^{|s|^{1+\varepsilon}} \cdot e^{|s|^{1+\varepsilon}} \;=\; e^{2\cdot|s|^{1+\varepsilon}} \;\le\; e^{|s|^{1+2\varepsilon}} \qquad\qquad (\text{for large } |s|)$$

Thus, in the half-plane $\mathrm{Re}(s) \ge \delta > 0$, the desired bound on $1/\Gamma(s)$ holds.

To get a bound in a left half-plane, say in $\mathrm{Re}(s) \le \frac{1}{2}$, use

$$\frac{1}{\Gamma(1-s)} \;=\; \Gamma(s)\cdot \pi \sin \pi s$$

The Stirling-Laplace asymptotic for $\Gamma(s)$ in $\mathrm{Re}(s) \ge \frac{1}{2}$ again gives $|\Gamma(s)| \ll_\varepsilon e^{|s|^{1+\varepsilon}}$ for all $\varepsilon > 0$. Certainly $\sin \pi s$ is of order 1, so, for $\mathrm{Re}(1-s) \le \frac{1}{2}$, we have $|1/\Gamma(1-s)| \ll_\varepsilon e^{|s|^{1+\varepsilon}}$ for all $\varepsilon > 0$. This completes the argument that $1/\Gamma(s)$ is entire of order 1. ///

[09.6] Let $d(n)$ be the *divisor function*, that is, the number of positive divisors of an integer n. Show that d is *weakly multiplicative* in the sense that $d(mn) = d(m)\cdot d(n)$ for m, n relatively prime, and that $d(p^\ell) = \ell + 1$ for p prime, and give some estimate on $d(n)$ adequate to show that $\sum_{n\ge 1} d(n)/n^s$ is absolutely convergent for $\mathrm{Re}(s)$ sufficiently large positive. Show that

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s} \;=\; \zeta(s)^2$$

Write $d|n$ when d divides n. To prove the first assertion, we prove that the positive divisors of mn for relatively prime m, n are exactly products dd', for positive divisors d of m and d' of n. One direction is easy: for $d|m$ and $d'|n$, certainly $dd'|mn$.

In the other direction, for $D|mn$, we claim that $D = \gcd(D, m) \cdot \gcd(D, n)$, where gcd is *greatest common divisor*. Since m, n are relatively prime, certainly the product of the gcd's divides D, but the other direction is a little subtler.

Just-in-case, we review some proof devices from elementary number theory. For example, *unique factorization* of the integers $\mathbb{Z}$ follows from the presence of the Euclidean algorithm, which shows that, given integers x, y, there are integers u, v such that $\gcd(x, y) = ux + vy$. Next, we claim that, for example, removing the greatest common divisor from two numbers leaves greatest common divisor just 1, that is, we claim that

$$\gcd\Big(\frac{x}{\gcd(x,y)}, \frac{y}{\gcd(x,y)}\Big) \;=\; 1$$

Indeed, dividing through $\gcd(x, y) = ux + vy$ by the gcd,

$$1 \;=\; u\frac{x}{\gcd(x,y)} + v\frac{y}{\gcd(x,y)}$$

Similarly, if $d|mn$ and d is coprime to m, necessarily $d|n$: write $1 = ud + vm$, and then

$$n \;=\; n \cdot 1 \;=\; n \cdot (ud + vm) \;=\; nud + nvm \;=\; d \cdot (nu + v\frac{mn}{d})$$

Thus, given $D|mn$, certainly $D/\gcd(D, m)$ divides $(m/gcd(D, m)) \cdot n$. But $D/\gcd(D, m)$ and $m/\gcd(D, m)$ are coprime, so $D/\gcd(D, m)$ divides n, and $D/\gcd(D, m)$ divides $\gcd(D, n)$. Then

$$\gcd(D,m)\cdot\gcd(D,n) \quad \text{divides} \quad D \;=\; \gcd(D,m)\cdot\frac{D}{\gcd(D,m)} \quad \text{divides} \quad \gcd(D,m)\cdot\gcd(D,n)$$

giving equality $D = \gcd(D, m) \cdot \gcd(D, n)$. Thus, set of divisors of a product mn of relatively prime m, n is in bijection with the cartesian product of the sets of divisors of m and of n, so $d(mn) = d(m)d(n)$. It is immediate that the divisors of a prime power p^ℓ are $1, p, p^2, \ldots, p^\ell$, giving $d(p^\ell) = \ell + 1$.

Certainly the number of positive divisors of n is no more than n itself, so $d(n) \le n$, and $\sum_n d(n)/n^s$ converges at least in $\mathrm{Re}(s) > 2$. In fact, we see that a stronger result holds, as follows.

Weak multiplicativity and unique factorization into prime powers gives the Euler product

$$\sum_n \frac{d(n)}{n^s} \;=\; \prod_{p \text{ prime}} \Big(1 + \frac{d(p)}{p^s} + \frac{d(p^2)}{(p^2)^s} + \ldots\Big) \;=\; \prod_{p \text{ prime}} \Big(1 + \frac{2}{p^s} + \frac{3}{(p^2)^s} + \ldots\Big)$$

In terms of $x = p^{-s}$, the power series

$$1 + 2x + 3x^2 + \ldots$$

is the derivative of $1 + x + x^2 + \ldots = 1/(1 - x)$, namely, $1/(1 - x)^2$. That is, for each p,

$$\Big(1 + \frac{2}{p^s} + \frac{3}{(p^2)^s} + \ldots\Big) \;=\; \Big(1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} + \ldots\Big)^2$$

and

$$\sum_n \frac{d(n)}{n^s} \;=\; \prod_{p \text{ prime}} \Big(1 + \frac{2}{p^s} + \frac{3}{(p^2)^s} + \ldots\Big) \;=\; \Big(\prod_{p \text{ prime}} \Big(1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} + \ldots\Big)\Big)^2 \;=\; \Big(\prod_p \frac{1}{1 - \frac{1}{p^s}}\Big)^2 \;=\; \zeta(s)^2$$

This makes possible derivation of some asymptotic results for $d(n)$. ///

[09.7] *(A variant Perron identity)* Show that, for $\sigma > 0$, a vertical path integral moving upward along the line $\mathrm{Re}(s) = \sigma$ evaluates to

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{X^s}{s(s+\theta)}\,ds \;=\; \begin{cases} \frac{1}{\theta}(1-X^{-\theta}) & (\text{for } X>1) \\ \\ 0 & (\text{for } 0<X<1) \end{cases} \qquad (\text{for } \theta>0,\ \sigma>0)$$

(The notation is the standard way of indicating a path integral over a vertical line.)

As with the more delicate Perron identity itself, the *idea* is that the vertical integration contour can be moved to the *right* for $0 < X < 1$, picking up no residues at all, giving 0, and moved to the *left* for $X > 1$, picking up the residues at the simple poles at $s = 0$ and $s = -\theta$. When this is justified, these residues are $X^0/(0+\theta)$ and $X^{-\theta}/(-\theta)$, together giving the indicated $(1 - X^{-\theta})/\theta$.

The issue is justification, which is easier in this case than for the original Perron identity, since here we have absolute convergence, unlike the original. That is, first, the indicated integral is absolutely convergent, and can be evaluated as a single limit (rather than worrying about the two tails as separate limits):

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{X^s}{s(s+\theta)}\,ds \;=\; \lim_{T\to+\infty}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{X^s}{s(s+\theta)}\,ds$$

For $0 < X < 1$, so that X^s decays to the *right*, view the finite vertical integral as the left side of an integral clockwise around the rectangle with vertices $\sigma - iT$, $\sigma + iT$, $T + iT$, $T - iT$, and then back to $\sigma - iT$. Noting that $|X^s| = X^{\mathrm{Re}(s)}$, the integrals over the three other sides are easily estimated: the top and bottom are

$$\Big|\int_{\sigma\pm iT}^{T\pm iT} \frac{X^s\,ds}{s(s+\theta)}\Big| \;\le\; \int_\sigma^T \frac{X^u\,du}{T^2} \;\le\; \int_\sigma^T \frac{1\,du}{T^2} \;\le\; \frac{1}{T} \;\longrightarrow\; 0$$

and the right side is

$$\Big|\int_{T+iT}^{T-iT} \frac{X^s\,ds}{s(s+\theta)}\Big| \;\le\; \int_{-T}^T \frac{X^{-T}\,du}{T^2} \;\le\; \int_{-T}^T \frac{1\,du}{T^2} \;\le\; \frac{2}{T} \;\longrightarrow\; 0$$

Thus, in the case $0 < X < 1$, the vertical integral is the negative (because the path integral is clockwise) sum of residues inside that rectangle, namely, 0.

For $X > 1$, so that X^s decays to the *left*, view the finite vertical integral as the right side of an integral counter-clockwise around the rectangle with vertices $\sigma - iT$, $\sigma + iT$, $-T + iT$, $-T - iT$, and then back to $\sigma - iT$. Much as in the case $0 < X < 1$, the integrals over the three other sides are easily estimated: the top and bottom are

$$\Big|\int_{\sigma\pm iT}^{-T\pm iT} \frac{X^s\,ds}{s(s+\theta)}\Big| \;\le\; \int_{-\sigma}^T \frac{X^{-u}\,du}{T^2} \;\le\; \int_{-\sigma}^T \frac{1\,du}{T^2} \;\le\; \frac{\sigma+T}{T^2} \;\longrightarrow\; 0$$

and the left side is

$$\Big|\int_{-T+iT}^{-T-iT} \frac{X^s\,ds}{s(s+\theta)}\Big| \;\le\; \int_{-T}^T \frac{X^{-T}\,du}{(T-\theta)^2} \;\le\; \int_{-T}^T \frac{1\,du}{(T-\theta)^2} \;\le\; \frac{2T}{(T-\theta)^2} \;\longrightarrow\; 0$$

Thus, again, the path integral around the rectangle captures the residues inside it, as indicated. ///

[09.8] In the Gaussian integers $\mathbb{Z}[i]$, there are 4 units $\pm1, \pm i$. The *norm* is $N(m + in) = m^2 + n^2$. Show that the zeta function

$$\zeta_{\mathbb{Z}(i)}(s) \;=\; \frac{1}{\#\mathbb{Z}[i]}\sum_{0\neq m+in\in\mathbb{Z}[i]} \frac{1}{N(m+in)^s} \;=\; \frac{1}{4}\sum_{m,n \text{ not both } 0} \frac{1}{(m^2+n^2)^s}$$

has an analytic continuation and functional equation

$$\pi^{-s}\Gamma(s)\zeta_{\mathbb{Z}[i]}(s) \;=\; \pi^{-(1-s)}\Gamma(1-s)\zeta_{\mathbb{Z}[i]}(1-s)$$

by using

$$\theta(y)^2 \;=\; \Big(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 y}\Big)^2 \;=\; \sum_{m,n\in\mathbb{Z}} e^{-\pi(m^2+n^2)y}$$

This is completely parallel to Riemann's argument for $\zeta(s)$, using a different but closely related *theta function*:

$$\Theta(y) \;=\; \theta(y)^2 \;=\; \Big(\sum_{m\in\mathbb{Z}} e^{-\pi n^2 y}\Big)^2 \;=\; \sum_{(m,n)\in\mathbb{Z}^2} e^{-\pi(m^2+n^2)y}$$

The theta function $\Theta(y)$ can also be considered directly, using the two-dimensional Fourier transform. The functional equation $\theta(1/y) = \sqrt{y}\cdot\theta(y)$ gives the functional equation

$$\Theta(1/y) \;=\; y\cdot\Theta(y)$$

As in Riemann's discussion, first we have the *integral representation* of $\zeta_{\mathbb{Z}[i]}(s)$ with its appropriate Gamma factor: for $\mathrm{Re}(s) > 1$,

$$\int_0^\infty y^s\,\frac{\Theta(y)-1}{4}\,\frac{dy}{y} \;=\; \frac{1}{4}\sum_{0\neq(m,n)\in\mathbb{Z}^2}\int_0^\infty y^s\, e^{-\pi(m^2+n^2)y}\,\frac{dy}{y}$$

$$=\; \pi^{-s}\cdot\frac{1}{4}\sum_{0\neq(m,n)\in\mathbb{Z}^2}\frac{1}{(m^2+n^2)^s}\int_0^\infty y^s\,e^{-y}\,\frac{dy}{y} \;=\; \pi^{-s}\Gamma(s)\cdot\frac{1}{4}\sum_{0\neq(m,n)\in\mathbb{Z}^2}\frac{1}{(m^2+n^2)^s} \;=\; \pi^{-s}\Gamma(s)\,\zeta_{\mathbb{Z}[i]}(s)$$

To prove the analytic continuation and functional equation, observe that the integral from 1 to ∞

$$\int_1^\infty y^s\,\frac{\Theta(y)-1}{4}\,\frac{dy}{y}$$

extends to an *entire* function, because the rapid decay of $\Theta(y)-1$ dominates the polynomial growth of y^s as $y\to+\infty$.

The integral from 0 to 1 can be converted to an integral from 1 to ∞ via the functional equation of Θ, with some leftover more-elementary terms: replacing y by $1/y$ and then rearranging gives

$$\int_0^1 y^s\,(\Theta(y)-1)\,\frac{dy}{y} \;=\; \int_1^\infty y^{-s}\,(\Theta(1/y)-1)\,\frac{dy}{y} \;=\; \int_1^\infty y^{-s}\,(y\Theta(y)-1)\,\frac{dy}{y}$$

$$=\; \int_1^\infty y^{1-s}\Big((\Theta(y)-1)+(1-\frac{1}{y})\Big)\,\frac{dy}{y} \;=\; \int_1^\infty y^{1-s}\,(\Theta(y)-1)\,\frac{dy}{y} + \int_1^\infty (y^{1-s}-y^{-s})\,\frac{dy}{y}$$

$$=\; \int_1^\infty y^{1-s}\,(\Theta(y)-1)\,\frac{dy}{y} + \frac{1}{s-1} - \frac{1}{s}$$

The integral converges very well for all $s\in\mathbb{C}$, so extends to an entire function, and the two rational functions extend to meromorphic functions on $\mathbb{C}$. Thus,

$$\pi^{-s}\Gamma(s)\,\zeta_{\mathbb{Z}[i]}(s) \;=\; \int_1^\infty (y^s+y^{1-s})\,\frac{\Theta(y)-1}{4}\,\frac{dy}{y} + \frac{1/4}{s-1} - \frac{1/4}{s} \qquad \text{(at first just for } \mathrm{Re}(s)>1\text{)}$$

presents $\pi^{-s}\Gamma(s)\,\zeta_{\mathbb{Z}[i]}(s)$ in a form exhibiting its meromorphic continuation and functional equation $s\longleftrightarrow 1-s$. ///