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Complex analysis examples discussion 10

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[This document is http://www.math.umn.edu/~garrett/m/complex/examples_2014-15/cx_discussion_10.pdf]

If you want feedback from me on your treatment of these examples, please get your work to me by Friday, Mar 27, preferably as a PDF emailed to me.

[10.1] Show that there is a well-defined, holomorphic function \( \frac{1}{\sqrt{1+z^4}} \) on the region \(|z| > 1\). Show that \( \int_\gamma \frac{dz}{\sqrt{1+z^4}} = 0 \), where \( \gamma \) traces out \(|z| = 2\).

There is a function \( \frac{1}{\sqrt{1+z^4}} \pm i \) well-defined on \(|z| > 1\), because

\[
\frac{1}{\sqrt{1+z^4}} = \frac{1}{z^2} \cdot \frac{1}{\sqrt{1 + \frac{1}{z^4}}} = \frac{1}{z^2} \cdot (1 + \frac{1}{z^4})^{-\frac{1}{2}} = \frac{1}{z^2} \cdot \left(1 - \frac{1}{2} \frac{1}{z^4} + \ldots\right)
\]

For \(|1/z^4| < 1\), the quantity \(1 + \frac{1}{z^4}\) stays in the right half-plane, so has a holomorphic square root throughout \(|z| > 1\). The Laurent expansion of the original function is then

\[
\frac{1}{\sqrt{1+z^4}} = \frac{1}{z^2} \cdot \frac{1}{\sqrt{1 + \frac{1}{z^4}}} = \frac{1}{z^2} \cdot (1 + \frac{1}{z^4})^{-\frac{1}{2}} = \frac{1}{z^2} \cdot \left(1 - \frac{1}{2} \frac{1}{z^4} + \ldots\right)
\]

By Cauchy’s theorem, the path integral of \(z^n\) around \(|z| = 2\) is 0 except for \(n = -1\), in which case it is \(2\pi i\). But there is no \(1/z\) term in that Laurent expansion. ///

[10.2] Let \(\gamma\) be a simple closed path counter-clockwise encircling 0, 2, and not enclosing \(-2\). Let \(\delta\) be a simple closed path counter-clockwise encircling \(-2, 0\), and not enclosing 2. Show that there is a holomorphic function \(1/\sqrt{z(z^2-4)}\) on the annulus \(1 < |z-1| < 3\), and a holomorphic function \(1/\sqrt{z(z^2-4)}\) on the annulus \(1 < |z+1| < 3\). Show that the two periods

\[
\int_\gamma \frac{dz}{\sqrt{z(z^2-4)}} \quad \int_\delta \frac{dz}{\sqrt{z(z^2-4)}}
\]

are linearly independent over \(\mathbb{R}\).

In fact, one is purely imaginary and the other is purely real. To show holomorphy in \(1 < |z-1| < 3\) and evaluate the integral around \(\gamma\), we determine (to some degree!) a Laurent expansion in that annulus. First,

\[
z(z^2-4) = ((z-1)+1)((z-1)-1)((z-1)+3) = 3(z-1)^2 \cdot \left(1 + \frac{1}{z-1}\right) \left(1 - \frac{1}{z-1}\right) \left(1 + \frac{z-1}{3}\right)
\]

Thus, the square root of the reciprocal is

\[
\frac{1}{\sqrt{3(z-1)}} \cdot \left(1 + \frac{1}{z-1}\right)^{-\frac{1}{2}} \cdot \left(1 - \frac{1}{z-1}\right)^{-\frac{1}{2}} \cdot \left(1 + \frac{z-1}{3}\right)^{-\frac{1}{2}}
\]

and although we cannot easily determine the coefficient of \((z-1)^{-1}\) in elementary terms, it is real, so the integral gives \(2\pi i\) times a real number.

Similarly, to obtain a Laurent expansion in the annulus \(1 < |z+1| < 3\),

\[
z(z^2-4) = ((z+1)-1)((z+1)+1)((z+1)-3) = -3(z+1)^2 \cdot \left(1 - \frac{1}{z+1}\right) \left(1 + \frac{1}{z+1}\right) \left(1 - \frac{z+1}{3}\right)
\]
Let \( \Lambda \) be a generality, the real interval \((1, \infty)\) for essentially elementary reasons. One approach is to deform the given contours to be Hankel/keyhole contours, as follows. Because the denominator is essentially of order \( R^{3/2} \) for large \( R = |z + 1| \), and of order \( r^{\frac{1}{2}} \) for small \( r = |z + 1| \), the path enclosing \(-2, 0 \) (and not enclosing \(+2\) ) can be deformed to an integral along a keyhole contour \( H^e \) from \(+2\) to \(+\infty\) with a small circle of radius \( \varepsilon > 0 \) about \( 2 \). Recall that for sufficiently small \( \varepsilon > 0 \) the value of the integral is independent of \( \varepsilon \). To match the outcome of the Laurent expansion, the integrand \( 1/\sqrt{z(z^2 - 4)} \) is required to take a purely imaginary value when the path crosses the real interval \((1, 2)\) at \( 2 - \varepsilon \). Thus, for continuity, the integrand is real on \((2, +\infty)\), and without loss of generality non-negative. The integral over the Hankel contour is

\[
\int_{H^e} \frac{dz}{\sqrt{z(z^2 - 4)}} = \frac{1}{1-e^{\pi i}} \int_2^\infty \frac{dt}{\sqrt{t(t^2 - 4)}} = \frac{1}{2} \int_2^\infty \frac{dt}{\sqrt{t(t^2 - 4)}} > 0
\]

A similar argument applies to prove that the other period is non-zero. Thus, since one is purely imaginary and the other purely real, they are linearly independent over \( \mathbb{R} \).

[10.3] Show that for irrational \( \alpha \in \mathbb{R} \), the set \( \{m + n\alpha : m, n \in \mathbb{Z}\} \) is dense in \( \mathbb{R} \).

(Kronecker) Let \( \Gamma \) be the topological closure of \( G = \mathbb{Z} + \mathbb{Z}\alpha \) in \( \mathbb{R} \). Suppose for a moment that we know the classification of all topologically-closed subgroups of \( \mathbb{R} \): either \( \{0\} \), \( \mathbb{R} \), or of the form \( \mathbb{R} \cdot \beta \) for some \( \beta \in \mathbb{R} \). The first case cannot occur for \( G \). If the last case occurs, then there are integers \( k, \ell \) such that \( k \cdot \beta = 1 \) and \( \ell \cdot \beta = \alpha \). But then \( \alpha = \ell/k \in \mathbb{Q} \), contradiction.

To prove the classification, for \( \Gamma \neq \{0\} \), closed under additive inverses, \( \Gamma \) contains positive elements. In the case that there is a least positive element \( \mu \), claim that \( \Gamma = \mathbb{Z} \cdot \mu \). Indeed, for \( \gamma \in \Gamma \), by the archimedean property of \( \mathbb{R} \) there is \( n \in \mathbb{Z} \) such that \( n\mu \leq \gamma < (n+1)\mu \). Necessarily \( n\mu = \gamma \), or else \( 0 < \gamma - n\mu < \mu \), contradicting the minimality.

In the case that there is not least positive \( \mu \), let \( \mu_1 > \mu_2 > \ldots > 0 \) be an infinite descending sequence of positive elements of \( \Gamma \). The inf \( \gamma_0 \) is in \( \Gamma \), since \( \Gamma \) is topologically closed. Replace \( \mu_n \) by \( \mu_n - \gamma_0 \) to be able to assume that \( \mu_n \to 0 \). Again by archimedean-ness, \( \mathbb{Z} \cdot \mu_n \) contains elements within \( \mu_n \) of every real number. Since \( \mu_n \to 0 \), for every \( \varepsilon > 0 \) \( \Gamma \) contains elements within \( \varepsilon \) of every real number. By closed-ness, \( \Gamma = \mathbb{R} \).

[10.4] Let \( v_1, \ldots, v_n \) be linearly independent vectors in \( \mathbb{R}^n \), and \( \Lambda = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_n \) the lattice generated by them. Let \( \mathbb{R}^n \) have its usual inner product and associated metric. For \( r > 0 \) let \( B_r \) be the ball of radius \( 0 \) centered at \( 0 \in \mathbb{R}^n \). Show that for small-enough \( r > 0 \) we have \( B_r \cap \Lambda = \{0\} \).

Let \( A \) be the invertible \( n \)-by-\( n \) real matrix so that \( Av_i = e_i \), where \( \{e_i\} \) is the standard basis of \( \mathbb{R}^n \). The map \( A : \mathbb{R}^n \to \mathbb{R}^n \) by \( v \to Av \) is continuous, and has continuous inverse given by multiplication by \( A^{-1} \). Thus, there is a small-enough neighborhood \( U \) of \( 0 \in \mathbb{R}^n \) such that the image \( AU \) is contained in the ball \( B_{\frac{1}{2}} \) of radius \( \frac{1}{2} \) centered at \( 0 \). Certainly \( B_{\frac{1}{2}} \cap \mathbb{Z}^n = \{0\} \). Then

\[
U \cap \Lambda \subset A^{-1}(B_{\frac{1}{2}} \cap \mathbb{Z}^n) = A^{-1}\{0\} = \{0\}
\]

Certainly \( U \) contains some ball at \( 0 \).

[10.5] Let \( \Lambda \) be a lattice in \( \mathbb{R}^n \), that is, the \( \mathbb{Z} \)-module generated by \( n \) vectors linearly independent over \( \mathbb{R} \). Prove that

\[
\sum_{0 \neq \lambda \in \Lambda} \frac{1}{|\lambda|^n}
\]
is absolutely convergent for $\text{Re}(s) > n$, where $|\cdot|$ is the usual length in $\mathbb{R}^n$. (Do not invoke any non-existent integral tests in several variables, despite the fact that the idea of such gives a good heuristic.)

Use volume to give a reasonable upper bound on the number of lattice points in shells $\ell \leq |x| \leq \ell + 1$. Namely, there is $0 < r < 1$ so that balls of radius $r$ centered at lattice points do not touch each other. If a lattice point $\lambda$ is in the shell $\ell \leq |x| \leq \ell + 1$, then the ball or radius $r$ around is is inside the thickened shell $\ell - r \leq |x| \leq \ell + r$, which is inside the shell $\ell - 1 \leq |x| \leq \ell + 1$. The latter has volume $C \cdot ((\ell + 1)^n - (\ell - 1)^n)$ for a constant $C$ depending on $n$. The total volume of all the balls of radius $r$ around lattice points is at most the volume of the thickened shell. Thus, the number of lattice points inside that shell has a good upper bound:

$$\# \text{ lattice points in } \{x \in \mathbb{R}^n : \ell \leq |x| \leq \ell + 1\} \leq \frac{C \cdot ((\ell + 1)^n - (\ell - 1)^n)}{C \cdot r^n} \leq C' \cdot \ell^{n-1} \quad \text{(for } \ell > 1)$$

for some constant $C'$. Similarly, for any radius $R$, the sum over $|\lambda| < R$ is finite: for such lattice points, the balls of radius $r$ around them are disjoint, and all lie inside the sphere of radius $R + r$ at 0, which has finite volume. Thus, to prove convergence, we can drop all $\lambda \in \Lambda$ with $|\lambda| \leq R$. Thus, for real $s > 0$ and $R = 3$, we can bound the lattice-point sum by a one-dimensional sum:

$$\sum_{\lambda \in \Lambda, |\lambda| \geq 3} \frac{1}{|\lambda|^s} = \sum_{\ell = 3}^{\infty} \sum_{\ell - 1 \leq |\lambda| < \ell + 1} \frac{1}{|\lambda|^s} \leq \sum_{\ell = 3}^{\infty} \sum_{\ell - 1 \leq |\lambda| < \ell + 1} \frac{1}{(\ell - 1)^s}$$

$$\leq C' \cdot \sum_{\ell = 3}^{\infty} \ell^{n-1} \frac{1}{(\ell - 1)^s} = C' \cdot \sum_{\ell = 2}^{\infty} (\ell + 1)^{n-1} \frac{1}{\ell^s} \leq C' \cdot 2^{n-1} \sum_{\ell = 2}^{\infty} \ell^{n-1} \frac{1}{\ell^s}$$

The usual one-dimensional integral test gives convergence of the latter for $s - (n - 1) > 1$, that is, for $s > n$.

///

[10.6] Recall that we need finite growth order $|f(z)| \ll e^{|z|^N}$ as $|z| \to +\infty$ in a strip $\alpha \leq \text{Re}(z) \leq b$, for some $N$, before we can invoke the Phragmén-Lindelöf theorem. Use the integral representation of $\zeta(s)$ via $\theta(y)$, and properties of $\Gamma(s)$, to show that it has finite order of growth in $-1 \leq \text{Re}(s) \leq 2$.

The integral representation

$$\pi^{-s} \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s) = \int_1^\infty (y^{s/2} + \frac{1}{\sqrt{\pi y}}) \frac{\theta(y) - 1}{2} \frac{dy}{y} + \frac{1}{s - 1} - \frac{1}{s}$$

(with $\theta(y) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 y}$)

converges absolutely for $s$ away from 0, 1. Thus, we can present $s(1 - s) \cdot \zeta(s)$ in a way that makes sense for all $s \in \mathbb{C}$:

$$s(1 - s) \cdot \zeta(s) = s(1 - s) \cdot \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}\right)} \cdot \left( \int_1^\infty (y^{s/2} + \frac{1}{\sqrt{\pi y}}) \frac{\theta(y) - 1}{2} \frac{dy}{y} + \frac{1}{s - 1} - \frac{1}{s} \right)$$

$$= \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}\right)} \cdot \left( s(1 - s) \int_1^\infty (y^{s/2} + \frac{1}{\sqrt{\pi y}}) \frac{\theta(y) - 1}{2} \frac{dy}{y} - s - (1 - s) \right)$$

The function $s \to \pi^{s/2}$ visibly has growth-order 1. Stirling’s asymptotic shows that in $\text{Re}(s) \geq \frac{1}{2}$ the function $s \to 1/\Gamma(s)$ has growth order 1, and the reflection relation $1/\Gamma(1 - s) = \frac{\sin \pi s}{\pi \Gamma(s)}$ yields the growth-order estimate in $\text{Re}(s) \leq \frac{1}{2}$. The integral is bounded in vertical strips of finite width, and the polynomials $s$ and $s - 1$ have growth order 0. Since the product and/or sum of functions of growth order $\alpha > 0$ is again such a function, this proves that $s(1 - s) \cdot \zeta(s)$ has growth order 1. (Phragmén-Lindelöf then gives a sharper assertion.)

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[10.7] Show that $f(x, y) = (x \pm iy)^{\ell} e^{-\pi(x^2 + y^2)}$ is multiplied by $i^{-\ell}$ by Fourier transform

$$\hat{f}(\xi, \eta) = \int_{\mathbb{R}^2} e^{-2\pi i (\xi x + \eta y)} f(x, y) \, dx \, dy$$

3
Hint: rewrite this in terms of $z = x + iy$ and $\overline{z}$, and another complex variable $w = \xi + i\eta$ and $\overline{w}$, and look for a chance to differentiate under the integral defining the Fourier transform.

Following the hint, and taking the plus sign, the Fourier transform is

$$
\hat{f}(w, \overline{w}) = \int_{\mathbb{R}^2} e^{-\pi i (\xi x + \eta y + \overline{w} z + \overline{\xi} x + \overline{\eta} y)} \, dx \, dy = \frac{1}{(-\pi i)^\ell} \int_{\mathbb{R}^2} \left( \frac{\partial}{\partial \overline{w}} \right)^\ell e^{-\pi i (\xi x + \eta y + \overline{\xi} x + \overline{\eta} y + \overline{w} z + \overline{\xi} x + \overline{\eta} y)} \, dx \, dy
$$

$$
= \frac{1}{(-\pi i)^\ell} \left( \frac{\partial}{\partial \overline{w}} \right)^\ell \int_{\mathbb{R}^2} e^{-\pi i (\xi x + \eta y + \overline{w} z + \overline{\xi} x + \overline{\eta} y)} \, dx \, dy = \frac{1}{(-\pi i)^\ell} \left( \frac{\partial}{\partial \overline{w}} \right)^\ell e^{-\pi w \overline{w}}
$$

since we already know that these pure Gaussians map to themselves under Fourier transform. Then this is

$$
\frac{1}{(-\pi i)^\ell}(-\pi w)^\ell \cdot e^{-\pi w \overline{w}} = i^{-\ell} \cdot w^\ell \cdot e^{-\pi w \overline{w}}
$$

as claimed. The argument for the minus-sign case is identical. //

[10.8] Define a harmonic theta function $\Theta_\ell(y)$ by

$$
\Theta_\ell(y) = \begin{cases} 
\frac{1}{4} \sum_{(m,n) \not\equiv (0,0) \in \mathbb{Z}^2} (m + in)^\ell e^{-\pi y (m^2 + n^2)} & \text{for } \ell > 0 \\
\frac{1}{4} \sum_{(m,n) \not\equiv (0,0) \in \mathbb{Z}^2} (m - in)^\ell e^{-\pi y (m^2 + n^2)} & \text{for } \ell < 0
\end{cases}
$$

Show that this is identically 0 unless $\ell$ is divisible by 4, and prove the functional equation

$$
\Theta_\ell(1/y) = y^{\ell+1} \cdot \Theta_\ell(y)
$$

Replacing $m \pm in$ by $i \cdot (m \pm in)$ is a bijection of non-zero Gaussian integers to themselves, so cannot alter the sum. Yet a factor of $i^\ell$ comes out, so the whole sum is multiplied by $i^\ell$. Thus, either the sum is 0, or $i^\ell = 1$.

In $\mathbb{R}^2$, by changing variables in the defining integral, $f(\sqrt{y} \cdot v) = \frac{1}{y} \hat{f}(\frac{1}{\sqrt{y}} \cdot v)$. Let $f(u, v) = (u \pm iv) e^{-\pi (u^2 + v^2)}$. Note that the term $m = n = 0$ in the sum defining the theta function vanishes for $\ell \neq 0$, and we consider only that situation. Also, take $\ell \in 4\mathbb{Z}$. The previous example’s computation of the Fourier transform of $(u \pm iv) e^{-\pi (u^2 + v^2)}$, Poisson summation, give

$$
\Theta(y) = \sum_{m,n \in \mathbb{Z}} (m \pm in)^\ell e^{-\pi y (m^2 + n^2)} = y^{-\ell/2} \sum_{m,n \in \mathbb{Z}} (\sqrt{y} (m \pm in))^\ell e^{-\pi y (m^2 + n^2)} = y^{-\ell/2} \sum_{m,n \in \mathbb{Z}} \hat{f}(\sqrt{y} \cdot (m,n))
$$

$$
= y^{-\ell/2} \frac{1}{y} \sum_{m,n \in \mathbb{Z}} \hat{f}(\frac{1}{\sqrt{y}} \cdot (m,n)) = y^{-\ell/2} \frac{1}{y} \sum_{m,n \in \mathbb{Z}} \left( \frac{1}{\sqrt{y}} (m \pm in) \right)^\ell e^{-\pi y (m^2 + n^2)}
$$

$$
= y^{-\ell/2} \frac{1}{y} \sum_{m,n \in \mathbb{Z}} (m \pm in)^\ell e^{-\pi y (m^2 + n^2)} = \frac{1}{y^{\ell+1}} \Theta(\frac{1}{y})
$$

as claimed. //

[10.9] Let $\chi(\alpha) = (\alpha/|\alpha|)^\ell$ for $\alpha \in \mathbb{C}^\times$. The associated Hecke $L$-function on the Gaussian integers $\mathbb{Z}[i]$ is

$$
L(s, \chi) = \frac{1}{\# \mathbb{Z}[i]} \sum_{\theta \neq \alpha \in \mathbb{Z}[i]} \frac{\chi(\alpha)}{|\alpha|^{2s}} = \frac{1}{4} \sum_{\theta \neq \alpha \in \mathbb{Z}[i]} \frac{\chi(\alpha)}{|\alpha|^{2s}}
$$
Show that this is identically 0 unless $\ell$ is divisible by 4. Prove that $L(s, \chi_\ell)$ has an analytic continuation and functional equation and has the integral representation

$$
\pi^{-s+|\ell|/2} \Gamma \left( s + \frac{|\ell|}{2} \right) L(s, \chi) = \int_0^\infty y^{s+|\ell|/2} \Theta_\ell(y) \frac{dy}{y} \quad \text{ (for Re}(s) > 1) $$

As with the theta functions in the previous, the change of variables by multiplying $m \pm in$ by $i$ merely permutes the summands, but multiplies the whole sum by $i^\ell$, so either the sum is 0 or $i^\ell = -1$. Thus, take $\ell \in 4\mathbb{Z}$. Also, the case $\ell = 0$ was treated earlier, so take $\ell \neq 0$.

As with Riemann’s zeta and other examples, these $L$-functions are Mellin transforms of the theta functions, once the normalizations are correctly determined. In the case $\ell > 0$,

$$
\int_0^\infty y^{s+\frac{\ell}{2}} \Theta_\ell(y) \frac{dy}{y} = \frac{1}{4} \sum_{0 \neq \alpha} \alpha^\ell \cdot \int_{0}^{\infty} y^{s+\frac{\ell}{2}} \cdot e^{-\pi y|\alpha|^2} \frac{dy}{y} = \frac{1}{4} \sum_{0 \neq \alpha} \alpha^\ell \cdot \int_{0}^{\infty} y^{s+\frac{\ell}{2}} \cdot e^{-\pi y|\alpha|^2} \frac{dy}{y} 
$$

$$
= \frac{\pi^{-s}}{4} \sum_{0 \neq \alpha} \alpha^\ell \cdot \int_{0}^{\infty} y^{s+\frac{\ell}{2}} \cdot e^{-\pi y|\alpha|^2} \frac{dy}{y} = \frac{1}{2} \sum_{0 \neq \alpha} \alpha^\ell \cdot \frac{1}{|s+\frac{\ell}{2}|} = \pi^{-s+\frac{|\ell|}{2}} \Gamma(s + \frac{\ell}{2}) \frac{\pi^{-s}}{4} \sum_{0 \neq \alpha} \alpha^\ell
$$

A parallel computation works for $\ell < 0$. Then break the integral into two pieces, from 0 to 1 and then 1 to $\infty$. The integral from 1 to $\infty$ converges nicely for all $s \in \mathbb{C}$, so gives an entire function. The integral from 0 to 1 is converted to one from 1 to $\infty$ by the change of variables $y \to 1/y$ and using the functional equation of $\Theta_\ell$:

$$
\int_0^1 y^{s+\frac{|\ell|}{2}} \Theta_\ell(y) \frac{dy}{y} = \int_1^\infty y^{-s-\frac{|\ell|}{2}} \Theta_\ell(1/y) \frac{dy}{y} = \int_1^\infty y^{-s-\frac{|\ell|}{2}} y^{|\ell|+1} \Theta_\ell(y) \frac{dy}{y} 
$$

$$
= \int_1^\infty y^{-s-\frac{|\ell|}{2}} \Theta_\ell(y) \frac{dy}{y}
$$

Thus,

$$
\pi^{-s+\frac{|\ell|}{2}} \Gamma(s + \frac{|\ell|}{2}) L(s, \chi_\ell) = \int_1^\infty (y^{s+\frac{|\ell|}{2}} + y^{-s-\frac{|\ell|}{2}}) \Theta_\ell(y) \frac{dy}{y}
$$

The right-hand side converges very well, so is entire. \hfill ///

[10.10] With $\chi_\ell(\alpha) = (\alpha/|\alpha|)^\ell$, and the $L$-functions $L(s, \chi)$ as in the previous example, express $L(4, \chi_{-8})$ as a polynomial in $L(2, \chi_{-4})$.

The specific arguments make these $L$-function values be essentially the **Eisenstein series** attached to the lattice of Gaussian integers:

$$
L(k, \chi_{-2k}) = \frac{1}{4} \sum_{0 \neq \lambda \in \mathbb{Z}[i]} \frac{(\lambda/|\lambda|)^{-2k}}{|\lambda|^2k} = \frac{1}{4} \sum_{0 \neq \lambda \in \mathbb{Z}[i]} \frac{\lambda^{2k}}{k} = \frac{1}{4} \cdot E_{2k}(\Lambda)
$$

One charming way to compare $E_4(\Lambda)$ and $E_8(\Lambda)$ is by using the rigid behavior of **modular forms**, which proves that up to constants there is a unique holomorphic elliptic modular form of weight 8 (for $SL_2(\mathbb{Z})$). Certainly $E_8$ is such, and $E_8^2$ is, also. Thus, $E_8$ is a constant multiple of $E_4^2$. With lattice $\Lambda = \mathbb{Z} + \mathbb{Z}$, in this normalization Eisenstein series have **Fourier expansions**

$$
E_{2k}(z) = 2\zeta(2k) + \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n \geq 1} \sigma_{2k-1}(n) e^{2\pi inz}
$$
Thus, making the leading coefficient 1,
\[ \left( \frac{E_4}{2\zeta(4)} \right)^2 = \frac{E_8}{2\zeta(8)} \]
so
\[ E_8 = E_4^2 \cdot \frac{\zeta(8)}{(4\zeta(4))^2} \]

Note that the constant is in fact rational.

\\

[10.11] Show how to achieve the effect of replacing a quartic by a cubic in an elliptic integral: exhibit a change of variables so that
\[ \int_a^b \frac{dx}{\sqrt{x^4 - 1}} = \int_A^B \frac{dy}{\sqrt{4y^4 + 6y^2 + 4y + 1}} \]
The trick is to use a linear fractional transformation to move one of the zeros of the quartic to \( \infty \). For example, \( g = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \) sends 1 to \( \infty \). Replacing \( x \) by \( g \cdot y \) gives
\[ \int \frac{d\left( \frac{-y}{y+1} \right)}{\sqrt{\left( \frac{-y}{y+1} \right)^4 - 1}} = \int \frac{1}{\left( \frac{-y}{y+1} \right)^2} \frac{dy}{\sqrt{\left( \frac{-y}{y+1} \right)^4 - 1}} = \int \frac{dy}{\sqrt{y^4 - (-y + 1)^4}} \]
Replace \( y \) further by \( y + 1 \), to obtain
\[ \int \frac{dy}{\sqrt{(y + 1)^4 - y^4}} = \int \frac{dy}{\sqrt{4y^4 + 6y^2 + 4y + 1}} \]
as desired.

\\

[10.12] Fix a lattice \( L \). Express
\[ f(z) = \frac{1}{z^4} + \sum_{0 \neq \lambda \in L} \frac{1}{(z - \lambda)^4} \]
in terms of \( \wp(z) \) and \( \wp'(z) \).

This function is even, so we anticipate it is expressible in terms of \( \wp(z) \). To obtain the expression, we try to cancel the poles, leaving an entire, doubly-periodic functions, which must be constant, by Liouville. The Laurent expansions of \( f \) and \( \wp \) at 0 are of the forms
\[ f(z) = \frac{1}{z^4} + a + bz^2 + \ldots \quad \wp(z) = \frac{1}{z^2} + Bz^2 + \ldots \]
since the convergence trick for \( \wp(z) \) makes the constant 0. In the proof of the Weierstrass equation, one discovers that \( B \) is essentially an Eisenstein series
\[ E_{2k}(\Lambda) = \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^{2k}} \]
Precisely, \( B = 3E_4 \). Similarly,
\[ a = \left( f(z) - \frac{1}{z^4} \right)_{z=0} = E_4 \]
Thus,
\[ \wp(z)^2 = \frac{1}{z^4} + 2B + O(z^2) \]
so
\[ f(z) - \varphi(z)^2 = (a - 2B) + O(z^2) \]
Thus, \[ f(z) - \varphi(z)^2 = a - 2B + O(z^2) \]
and by Liouville \[ f(z) - \varphi(z)^2 = a - 2B = E_4 - 6E_4 = -5E_4 \]

[10.13] Express \( \varphi(2z) \) in terms of \( \varphi(z) \).

Let \( \omega_1, \omega_2 \) be a basis for the lattice \( \Lambda \). Since \( \varphi(2z) \) has double poles at \( \Lambda/2 \), with residues \( 1/4 \), \( \varphi(2z) - \frac{1}{4}\varphi(z) \) has double poles exactly at \( \frac{\omega_1}{2} + \Lambda, \frac{\omega_2}{2} + \Lambda \), and \( \frac{\omega_1 + \omega_2}{2} + \Lambda \).

Let \( a \) be any one of \( \omega_1/2, \omega_2/2, \) or \( (\omega_1 + \omega_2)/2 \). Since \( \varphi(z) - \varphi(a) \) is still even, and has a zero at \( z = a \), this is a double zero. Since the number of poles is equal to the number of zeros, and \( \varphi(z) \) has a double pole, there are no other zeros of \( \varphi(z) - \varphi(a) \). Thus,
\[
(\varphi(2z) - \frac{1}{4}\varphi(z)) \cdot ((\varphi(z) - \varphi(\frac{\omega_1}{2}))(\varphi(z) - \varphi(\frac{\omega_2}{2}))(\varphi(z) - \varphi(\frac{\omega_1 + \omega_2}{2})))
\]
is an even function that has poles only on the lattice.

Again, the Laurent expansion of \( \varphi(z) \) is the power series of \( \varphi(z) - 1/z^2 \) plus \( 1/z^2 \), and the coefficients of the power series can be computed via derivatives, giving
\[
\frac{1}{z^2} + 3E_4z^2 + 5E_6z^4 + 7E_8z^6 + \ldots
\]
Thus,
\[
\varphi(2z) - \frac{1}{4}\varphi(z) = 3E_4(2^2 - 1)z^2 + 5E_6(2^3 - 1)z^4 + 7E_8(2^8 - 1)z^6 + \ldots
\]
Thus, the (at least) double zero partly cancels the order-six pole, and the order of pole of the adjusted function is at most 4. Via Liouville’s theorem, it is inevitably a polynomial in \( \varphi(z) \), of degree at most 2:
\[
(\varphi(2z) - \frac{1}{4}\varphi(z)) \cdot ((\varphi(z) - \varphi(\frac{\omega_1}{2}))(\varphi(z) - \varphi(\frac{\omega_2}{2}))(\varphi(z) - \varphi(\frac{\omega_1 + \omega_2}{2}))) = A\varphi(z)^2 + B\varphi(z) + C
\]
This leaves 3 constants \( A, B, C \) to be determined. The leading \( 1/z^4 \) coefficient of the Laurent expansion is the \( z^2 \) coefficient of \( \varphi(2z) - \varphi(z)/4 \), namely, \( 9E_4 \). This is the constant \( A \), so
\[
(\varphi(2z) - \frac{1}{4}\varphi(z)) \cdot ((\varphi(z) - \varphi(\frac{\omega_1}{2}))(\varphi(z) - \varphi(\frac{\omega_2}{2}))(\varphi(z) - \varphi(\frac{\omega_1 + \omega_2}{2}))) - 9E_4\varphi(z)^2 = B\varphi(z) + C
\]
Evaluating the equation at a zero \( z_o \) of \( \varphi(z) \) gives
\[
-\varphi(2z_o) \cdot \varphi(\frac{\omega_1}{2}) \cdot \varphi(\frac{\omega_2}{2}) \cdot \varphi(\frac{\omega_1 + \omega_2}{2}) = C
\]
If \( \varphi(2z_o) \) has a pole at \( z = z_o \), then \( z_o \) is among the 2-division-point values \( \omega_1/2, \omega_2/2, (\omega_1 + \omega_2)/2 \), and this evaluation requires a little more effort (which we do not exert here).

To determine \( B \), we could take a derivative of the equation and evaluate anywhere \( \varphi'(z) \neq 0 \), ...

If we cared more, we could pursue this or various other possibilities, such as multiplying out the Laurent expansions at 0 and comparing terms.
Show that
\[ \theta(z) = \sum_{v \in \mathbb{Z}^8} e^{\pi i |v|^2 \cdot z} \quad (\text{with } z \in \mathfrak{F}) \]
is an elliptic modular form of weight 4 for the congruence subgroup \( \Gamma_\theta \).

We need to use the fact that the subgroup \( \Gamma_\theta \) is generated by \( z \to z + 2 \) and \( z \to -1/z \). The invariance of \( \theta(z) \) under \( z \to z + 2 \) is clear, since each term is unchanged. Taking \( z = iy \) with \( y > 0 \), Poisson summation proves \( \theta(iy) = \frac{1}{(iy)^4} \theta(i/y) \). By the identity principle, \( \theta(-1/z) = z^4 \cdot \theta(z) \), which is the correct behavior for a weight-four modular form.