(*November 7, 2014*)

Complex analysis examples 06

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[This document is http://www.math.umn.edu/˜garrett/m/complex/examples_2014-15/cx_ex_06.pdf]

If you want feedback from me on your treatment of these examples, please get your work to me by Friday, Nov 21, preferably as a PDF emailed to me.

[06.1] Evaluate $\int_0^{2\pi} \frac{dt}{2+\cos t}$

[06.2] Show that $z^{10} - z^7 + 4z^2 - 1 = 0$ has exactly two zeros inside the circle $|z| = 1$.

[06.3] Show that $\cos z$ has exactly two complex zeros inside $|z| = 2$ by comparing it to $1 - \frac{z^2}{2}$, which certainly has exactly two complex zeros inside that circle.

[06.4] Prove that, given holomorphic f, g on a non-empty open set U, and given a *simple* zero z_o of f, for all small-enough complex ε the zero of $f + \varepsilon g$ nearest z_o is also *simple*.

[06.5] Let U be the region

$$U \;=\; \{z \in \mathbb{C} \;:\; \mathrm{Re}(z) > 0,\; |z - (1+i)| > 1,\; |z - (1-i)| > 1\}$$

Let $\widetilde{U}$ be the topological closure of U with the point 1 removed. (Thus, $\widetilde{U}$ includes the interval $[-i, +i]$ along the imaginary axis, and two quarter-circles with the point 1 removed.) Construct a holomorphic function on U extending to a continuous function on $\widetilde{U}$, bounded by 1 on the boundary *except for* 1, but *unbounded* on U.

[06.6] Let C be the usual Cantor set

$$C \;=\; \{x \in [0,1] : \text{the ternary expansion of } x \text{ contains only digits 0 and 2, digit 1}\}$$

where terminal repeating 1's $(\ldots 111111 \ldots)$ are converted to $\ldots 2$. Show that there is no non-constant holomorphic function with real part taking values in C.

[06.7] Given $R > 0$, $w_o \in \mathbb{C}$, and $\varepsilon > 0$, show that there is $z \in \mathbb{C}$ with $|z| > R$ and $|e^z - w_o| < \varepsilon$.

[06.8] For small $w \in \mathbb{C}$, let $f(w)$ be the simple zero of $z^5 - z + w = 0$ near 0. Determine a few terms of the power series expansion of $f(w)$ at $w = 0$.
